CONSTRUCTION OF SOLUTIONS TO THE EINSTEIN CONSTRAINT EQUATIONS IN GENERAL RELATIVITY AND COMMENTS ON THE POSITIVE MASS THEOREM

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CONSTRUCTION DE SOLUTIONS POUR LES ÉQUATIONS DE CONTRAINTES EN RELATIVITÉ GÉNÉRALE ET REMARQUES SUR LE THÉORÈME DE LA MASSE POSITIVE

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Résumé

Dans cette thèse nous étudions deux problèmes issus de la relativité générale : la construction de données initiales pour le problème de Cauchy des équations d'Einstein et le théorème de la masse positive.

Nous construisons tout d'abord des données initiales en utilisant la méthode dite conforme introduite par Lichnerowicz [Lichnerowicz, 1944], Y. Choquet-Bruhat–J. York [Choquet-Bruhat et York, 1980] et Y. Choquet-Bruhat–J. Isenberg– D. Pollack [Choquet-Bruhat *et al.*, 2007a]. Plus particulièrement, nous étudions les équations –de contrainte conforme– qui apparaissent dans cette méthode sur des variétés riemanniennes compactes de dimension $n \geqslant 3$. Dans cette thèse, nous donnons une preuve simplifiée du résultat de [Dahl *et al.*, 2012], puis nous étendons et nous généralisons les théorèmes de M. Holst–G. Nagy–G. Tsogtgerel [Holst *et al.*, 2009] et de D. Maxwell [Maxwell, 2009] dans le cas de données initiales à courbure moyenne fortement non-constante. Nous donnons au passage un point de vue unifié sur ces résultats. En parallèle, nous donnons des résultats de non-existence et de non-unicité pour les équations de la méthode conforme sous certaines hypothèses.

Pour le second problème, nous étudions le théorème de la masse positive sur des variétés asymptotiquement hyperboliques. Plus précisément, nous montrons que la positivité de la masse sur une variété asymptotiquement hyperbolique est préservée par les chirurgies de codimension au moins 3. Par conséquent, nous étendons l'un des résultat principaux de l'article d'E. Humbert et A. Hermann [Humbert et Herman, 2014] aux variétés asymptotiquement hyperboliques en montrant que le théorème de la masse positive est vrai sur les variétés asymptotiquement hyperboliques de dimension $n \geqslant 5$ pourvu qu'il le soit pour une seule variété asymptotiquement hyperbolique simplement connexe non spin de dimension n.

Cette thèse ne requière aucune connaissance particulière hormis un bagage minimal en analyse géométrique.

Mots clés : équations d'Einstein, méthode dite conforme, théorème de la masse positive

Abstract

The aim of this thesis is the study of two topical issues arising from general relativity: finding initial data for the Cauchy problem with respect to the Einstein equations and the positive mass theorem.

For the first issue, in the context of the conformal method introduced by Lichnerowicz [Lichnerowicz, 1944], Y. Choquet-Bruhat–J. York [Choquet-Bruhat et York, 1980] and Y. Choquet-Bruhat–J. Isenberg–D. Pollack [Choquet-Bruhat *et al.*, 2007a], we consider *the conformal constraint equations* on compact Riemannian manifolds of dimension $n \geqslant 3$. In this thesis, we simplify the proof of [Dahl *et al.*, 2012, Theorem 1.1], extend and sharpen the far-from CMC result proven by Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009], Maxwell [Maxwell, 2009] and give an unifying viewpoint of these results. Besides discussing the solvability of the conformal constraint equations, we will also show nonexistence and nonuniqueness results for solutions to the conformal constraint equations under certain assumptions.

For the second one, we are interested in studying the positive mass theorem on asymptotically hyperbolic manifolds. More precisely, we prove that positivity of the mass of an asymptotically hyperbolic manifold is kept under a finite sequence of surgeries of codimension at least 3. As a consequence, we extend one of main results of Humbert–Hermann [Humbert et Herman, 2014, Theorem 8.5] to asymptotically hyperbolic manifolds, that is the positivity of the mass holds on all asymptotically hyperbolic manifolds of dimension $n \geqslant 5$, provided it is so on a single simply connected non-spin asymptotically hyperbolic manifold of the same dimension.

This thesis is mainly self-contained except for minimum background in Geometric Analysis.

Keywords : Einstein constraint equations, non-constant mean curvature, conformal method, positive mass theorem.

Contents

General Introduction

Albert Einstein introduced his theories of special relativity in 1905 and of general relativity in 1915. Both these theories were more breakthroughs since they reconsider the intuitive idea of space and time as two distinct unalterable objects. Special relativity originated as a mean to reconcile Newton's theory of motion with the Galilean principle and Maxwell's theory of electromagnetism. Its main idea may look naive yet extremely profound: if every inertial observer is to measure the same speed for light independently of his velocity with respect to any other observer, this means that the way that distances and/or time are measured depends on the observer.

Hermann Minkowski interpreted the coordinate transformations corresponding to a change of inertial observer introduced by Hendrik Lorentz and that form the ground of special relativity as isometrics of $\mathbb{R}^{3+1}$ endowed with the quadratic form

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2.$$

As we saw, special relativity was proposed as a natural framework for electromagnetism. But soon after the question of introducing other forces arose, in particular, the gravitational force, which was the only other fundamental force known at that time. It took eight years (from 1907 to 1915) for Einstein to develop a coherent theory. General relativity stands on the following –equivalence– principle. The inertial mass, which makes heavy objects harder to move than light objects, is equal to the gravitational mass, namely the strength with which an object interacts with the gravitational field. Gravity shares this property with the non-inertial forces such as the centrifugal force so it is possible to find non-inertial frames where the gravitational field is (almost) zero such as the "zero-gravity" flight or to create artificial gravity in accelerated reference frames as Tintin rocket or fighter planes.

Thus, general relativity has to do with arbitrary changes of coordinate system and hence is an adaptation of the then newly born theory of Riemannian geometry.

More precisely, in general relativity, a space-time is a $(n+1)-$dimensional Lorentzian manifold $(\mathcal{M}, h)$ (Lorentzian meaning that h has signature $- \; + \; + \; ... \; +$), with $n \geqslant 3$, which satisfies the Einstein equations

$$\mathrm{Ric}^h_{\mu\nu} - \frac{\mathrm{Scal}_h}{2} h_{\mu\nu} = \frac{8\pi\mathcal{G}}{c^4} T_{\mu\nu}, \tag{1}$$

where Ric^h, Scal_h are respectively Ricci and the scalar curvatures of of h, $\mathcal{G}$ is Newton's constant, c is the speed of light and T is the stress-energy tensor of non-gravitational fields (i.e., matter fields, electromagnetic field...).

Einstein equations are roughly speaking hyperbolic. Hence all solutions can be obtained from their initial values at some "time t=0", the metric $\hat{g}$ induced on a spacelike hypersurface $M \subset \mathcal{M}$, and its initial velocity, the second fundamental form $\hat{K}$ of the embedding $M \subset \mathcal{M}$. It follows from (1) and by the Gauss and Codazzi equations that the choice of $(M, \hat{g}, \hat{K})$ must satisfy the so-called Einstein constraint equations. In a scalar-field theory; e.g, the expression of T involves a scalar field Ψ and a potential V, namely,

$$T_{\mu\nu} = \nabla_\mu \Psi \nabla_\nu \Psi - \left(\frac{1}{2}|d\Psi|_h^2 + V(\Psi)\right) h_{\mu\nu},$$

the constraint equations read

$$\begin{aligned} \mathrm{Scal}_{\hat{g}} - |\hat{K}|_{\hat{g}}^2 + \left(\mathrm{tr}_{\hat{g}}\hat{K}\right)^2 &= \hat{\pi}^2 + |d\hat{\psi}|_g^2 + 2V(\hat{\psi}), \\ \mathrm{div}^{\hat{g}}\hat{K} - d\left(\mathrm{tr}_{\hat{g}}\hat{K}\right) &= \hat{\pi} d\hat{\psi}, \end{aligned} \tag{2}$$

where $\hat{\psi} = \Psi_{|M}$ is the restriction of Ψ to M and $\hat{\pi} = \nabla_\nu \Psi$ is the time derivative of Ψ. This system is the so-called Einstein-scalar field equations.

Constructing and classifying solutions to this system is an important issue. For a deeper discussion of (2) , we refer the reader to the excellent review article [Bartnik et Isenberg, 2004]. One of the most efficient methods to find initial data satisfying (2) is the conformal method developed by Lichnerowicz [Lichnerowicz, 1944] and Y. Choquet-Bruhat–Jr. York [Choquet-Bruhat et York, 1980]. The idea of this method is to effectively parameterize the solutions to (2) by some reasonable parts and then solve for the rest of the data. More precisely, we assume given some seed data: a compact Riemannian manifold (M, g), a mean curvature τ, a transverse-traceless tensor σ (i.e., a symmetric, trace-free, divergence-free $(0, 2)$-tensor), two functions ψ, π and a potential V. Then we look for a positive function φ and a $1-$form W such that

$$\hat{g} = \varphi^{N-2} g, \quad \hat{K} = \frac{\tau}{n}\varphi^{N-2} g + \varphi^{-2}(\sigma + LW), \quad \hat{\psi} = \psi, \quad \hat{\pi} = \varphi^{-N}\pi$$

is a solution to the Einstein-scalar field constraint equations (2). Here $N = \frac{2n}{n-2}$ and L is the conformal Killing operator defined by

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}(\mathrm{div} W) g_{ij}.$$

Equations (2) are now reformulated into the following coupled nonlinear elliptic system for φ and W:

$$\frac{4(n-1)}{n-2}\Delta\varphi + \mathcal{R}_\psi \varphi = \mathcal{B}_{\tau,\psi}\varphi^{N-1} + \left(|\sigma + LW|^2 + \pi^2\right)\varphi^{-N-1} \quad \text{[Lichnerowicz equation]} \tag{3a}$$

$$-\frac{1}{2}L^* LW = \frac{n-1}{n}\varphi^N d\tau - \pi d\psi, \quad \text{[vector equation]}, \tag{3b}$$

where Δ is the nonnegative Laplace operator, $L^*$is the formal L^2-adjoint of L and $\mathcal{R}_\psi$, $\mathcal{B}_{\tau,\psi}$ are given by

$$\mathcal{R}_\psi = \mathrm{Scal}^g - |d\psi|_g^2, \quad \mathcal{B}_{\tau,\psi} = -\frac{n-1}{n}\tau^2 + 2V(\psi).$$

These coupled equations are called the *conformal constraint equations*. Before going to further statements about the system (3), we give standard conditions for initial data and introduce some notations used in this thesis as follows.

Initial data. Let M be a compact manifold of dimension n with $n \geqslant 3$. Our goal in this thesis is to find solutions to the conformal constraint equations (3). The given data on M consists of

- a Riemannian metric $g \in C^2$
- a scalar field function $\psi \in W^{1,p}$ and its potential $V \in C^\infty(\mathbb{R})$,
- a function $\pi \in L^p$, (4)
- a function $\tau \in W^{1,p}$,
- a symmetric, trace- and divergence-free $(0,2)-$tensor $\sigma \in W^{1,p}$,

with $p > n$, and one is required to find

- a positive function $\varphi \in W^{2,\frac{p}{2}}$,
- a $1-$form $W \in W^{2,\frac{p}{2}}$,

which satisfy the conformal constraint equations (3). We also assume that

- (M, g) has no conformal Killing vector field,
- $(\sigma, \pi) \not\equiv (0,0)$ if $\mathcal{Y}_g \geqslant 0$, (5)

where $\mathcal{Y}_g$ is the modified Yamabe constant of the conformal class of g; that is

$$\mathcal{Y}_g = \inf_{\substack{f \in C^\infty(M) \\ f \not\equiv 0}} \frac{\frac{4(n-1)}{n-2} \int_M |\nabla f|^2 dv + \int_M \left(\mathrm{Scal} - |d\psi|_g^2\right) f^2}{\|f\|^2_{L^N(M)}}.$$

We use standard notations for function spaces, such as L^p, C^k, and Sobolev spaces $W^{k,p}$. It will be clear from the context if the notation refers to a space of functions on M, or a space of sections of some bundle over M. For spaces of functions which embed into L^∞, the subscript $+$ is used to indicate the cone of positive functions.

We will sometimes write, for instance, $C(\alpha_1, \alpha_2)$ to indicate that a constant C depends only on α_1 and α_2.

Since the treatment of the system (3) depends on the sign of $\mathcal{B}_{\tau,\psi}$, we divide our discussion into two cases.

- **The vacuum case:** During the past decades, existence of solutions to (3) in the vacuum case; i.e., $\psi \equiv \pi \equiv 0$ and $V \equiv 0$, was extensively studied. When the mean curvature τ is constant, the system (3) becomes uncoupled (since $d\tau = 0$ in the vector equation) and a complete description of the situation was achieved by J. Isenberg [Isenberg, 1995]. The near CMC case (i.e., when $d\tau$ is small) was addressed soon after. Most results can be found in [Bartnik et Isenberg, 2004]. For arbitrary τ however, the situation appears much harder and only two methods exist to tackle this case:

- (Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009] and Maxwell [Maxwell, 2009]) Assume that (M, g) has positive Yamabe invariant. Then for any τ, there exists an $\epsilon(g, \tau) > 0$ such that if
$$0 < \max |\sigma| < \epsilon \tag{6}$$
the system (3) has (at least) one solution.
- (Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012]) If τ has constant sign and if the *limit equation*
$$-\frac{1}{2}L^*LV = \alpha\sqrt{\frac{n-1}{n}}|LV|\frac{d\tau}{\tau} \tag{7}$$
has no non-zero solution, for all values of the parameter $\alpha \in [0, 1]$, then the set of solutions (φ, W) to (3) is nonempty and compact. This criterion holds true e.g. when (M, g) has $\mathrm{Ric} \leqslant -(n-1)g$ and $\left\|\frac{d\tau}{\tau}\right\|_{L^\infty} < \sqrt{n}$. It is worth noting that this result was successfully extended to asymptotically hyperbolic manifolds by Gicquaud–Sakovich [Gicquaud et Sakovich, 2012].

Conversely, nonexistence and nonuniqueness results for (3) are fairly rare for non-constant τ. We refer to arguments of Rendall, as presented in [Isenberg et Ó Murchadha, 2004], Holst–Meier [Holst et Meier, 2012], and Dahl–Gicquaud-Humbert [Dahl *et al.*, 2013] for attempts to obtain such results. In the vacuum case, the only model of nonuniqueness of solutions is constructed on the $n-$torus by D. Maxwell [Maxwell, 2011] while the only nonexistence result, achieved by J. Isenberg–Murchadha [Isenberg et Ó Murchadha, 2004] and later strengthened in [Dahl *et al.*, 2012] and [Gicquaud et Ngô, 2014], states that the system (3) with $\sigma \equiv 0$ has no solution when $\mathcal{Y}_g \geqslant 0$ and $d\tau/\tau$ is small enough. This assertion together with experimentations on the torus led D. Maxwell to ask whether the non-zero assumption of σ is a necessary condition for existence of solution to the conformal equations (3) with positive Yamabe invariant (see [Maxwell, 2011]).

From the results above, one may ask different questions:

1.1 Is the condition (6) sharp?

1.2 What role does the sign of τ play in the arguments of Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012]?

1.3 Can the parameter α in the limit equation (7) be assumed to be 1?

1.4 Is the system of (3) solvable for all $\sigma \not\equiv 0$? If a solution to (3) exists, is it unique?

1.5 What is the answer to Maxwell's question?

After briefly sketching basic facts on the Lichnerowicz equations in Chapter 1, which include the special case of (3) when τ is constant, these questions will be addressed in the next two chapters. The following is a summary.

Chapter 2:

We first introduce two new methods for solving (3) in Chapter 2. The first relies on the Leray-Schauder fixed point theorem while the second is based on the concept of half-continuity. These methods are used as main tools for addressing properties of solutions to (3) along the thesis. In particular, we will show in Chapter 2 that they not only simplify two far-from CMC results of Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009] and Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012] respectively, but also give unifying viewpoint of them. Furthermore, we obtain the following interesting results: one is a sharping of the smallness assumption of σ in (6) and the other shows that the principal estimate in arguments of Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012] for getting a non-trivial solution to (7) may fail when τ changes sign, and this gives an answer to our first two questions.

Theorem 1 (*Small TT-tensor*)**.** *Let data be given on M as specified in (4) associated to the vacuum case and assume that conditions (5) hold. Assume further that the Yamabe invariant $\mathcal{Y}_g > 0$. Then there exists $\epsilon(g,\tau) > 0$ such that if*

$$0 < \int_M |\sigma|^2 dv < \epsilon, \tag{8}$$

the system (3) *has (at least) one solution.*

Proposition 2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geqslant 3$. Suppose $g \in C^2$ and $\tau \in C^0$. For given $\alpha \in [1/N, +\infty]$, we denote*

$$A_\alpha = \sup_{(\varphi,W)\in\mathcal{L}} \frac{\|\varphi\|^N_{L^{N\alpha}}}{\max\{\|LW\|_\infty, 1\}},$$

where

$$\mathcal{L} = \left\{(\varphi, W) \in W^{2,p}_+ \times L^\infty :\ (\varphi, W) \text{ satisfies the vacuum Lichnerowicz equation } (3a)\right\}.$$

Then A_α is bounded if and only if $|\tau|^{-\alpha} \in L^1$.

Chapter 3:

Chapter 3 may be understood as a continuation of the previous one. More precisely, we will continue using both methods introduced in Chapter 2 to try to address all remaining questions. In fact, we show that:

Theorem 3 (*Control of the parameter*)**.** *Let data be given on M as specified in (4) associated to the vacuum case, and assume that conditions (5) hold. If τ has constant sign, then at least one of the following assertions is true* *(i)* *The conformal constraint equations* (3) *admit a solution* (φ, W) *with* $\varphi > 0$*. Furthermore, the set of solutions* $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$*, with* $p > n$*, is compact.*

(ii) *There exists a nontrivial solution* $V \in W^{2,p}$ *to the limit equation*

$$-\frac{1}{2}L^*LV = \sqrt{\frac{n-1}{n}}|LV|\frac{d\tau}{\tau}. \tag{9}$$

(iii) *For all continuous functions* $f > 0$ *or* $f \equiv R$ *if* $\mathcal{Y}_g > 0$*, the (modified) conformal constraint equations*

$$\frac{4(n-1)}{n-2}\Delta\varphi + f\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |LW|^2\varphi^{-N-1} \tag{10a}$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N d\tau \tag{10b}$$

have a (non-trivial) solution $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$*. Moreover if the corresponding Yamabe invariant* $\mathcal{Y}_g > 0$*, then there exists a sequence* $\{t_i\}$ *converging to* 0 *s.t. the conformal constraint equations* (3) *associated to seed data* $(g, t_i\tau, \sigma)$ *have at least two solutions.*

Comparing with the original version of Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012], the price to pay to set the parameter $\alpha = 1$ in (7) is the addition of (*iii*). We will see that this assertion is necessary because of the following result.

Theorem 4 (*Nonexistence of solution*)**.** *Let data be given on* M *as specified in* (4) *associated to the vacuum case, and assume that conditions* (5) *hold. Further assume that* τ *has constant sign and that there exists* $c = c(g) > 0$ *s.t.* $\left|L\left(\frac{d\tau}{\tau}\right)\right| \leqslant c\left|\frac{d\tau}{\tau}\right|^2$*. Let* U *be a given open neighborhood of the critical set of* τ*. If* $\sigma \not\equiv 0$ *and supp*$\{\sigma\} \subsetneq M \setminus U$*, then both of the conformal constraint equations* (3) *and the limit equation* (9) *associated to the seed data* $(g, \tau^a, k\sigma)$ *admit no (nontrivial) solution, provided* $a,\ k$ *are large enough.*

It is worth noting that [Dahl *et al.*, 2012, Proposition 1.6] provides the existence of such assumptions. In fact, our proof for Theorem 4 is an extension of arguments in [Dahl *et al.*, 2012, Proposition 1.6].

As direct consequences of Theorem 3 and 4, we also obtain the following results.

Corollary 5 (*Nonuniqueness of solutions*)**.** *Assume that* $(M, g, \tau, \sigma, a, k)$ *is given as in Theorem* 4*. If* $\mathcal{Y}_g > 0$*, then there exists a sequence* $\{t_i\}$ *converging to* 0 *s.t. the conformal constraint equations* (3) *associated to seed data* $(g, t_i\tau^a, k\sigma)$ *have at least two solutions.*

Corollary 6 (*An answer to Maxwell's question*)**.** *Let (M, g, τ) be given as in Theorem 4. If $\mathcal{Y}_g > 0$, then the conformal constraint equations* (3) *associated to $(g, \tau^a, 0)$ have a (nontrivial) solution for all $a > 0$ large enough.*

- **The scalar field case:** In the presence of a scalar field case; i.e., for a general (ψ, V, π), the system (3) becomes much more difficult because $\mathcal{B}_{\tau,\psi}$ may become positive in the Lichnerowicz equation. This appears to be problematic since one loses the maximum principle. It makes the methods used in the vacuum cases above not working any longer, and then seems to tell us to seek different approaches for this case. Thus, there are less results in this situation. Perhaps, the most natural approach for the problem is to try to extend known results in the vacuum case to the scalar field one. However, not all of them are true in full generality, for instance, we will see in Chapter 1 that the Lichnerowicz equation, understood as the CMC case, can admit zero or several solutions if $\mathcal{B}_{\psi,\tau} > 0$. For deeper discussions of the problem, we refer the reader to Choquet-Bruhat–Isenberg–Pollack [Choquet-Bruhat *et al.*, 2007a], Hebey-Veronelli [Hebey et Veronelli, 2014] and Chruściel-Gicquaud [Chruściel et Gicquaud,] for the CMC case and to Premoselli [Premoselli, 2014] for the near-CMC case. Until now all results remained limited to the near-CMC case.

In Chapters 4 and 5, we will turn our attention to the properties of solutions for the far-from CMC case. More precisely, we will extend Theorem 1 to the scalar field case and show a nonuniqueness result for solutions to the (actual) system (3) as $\mathcal{B}_{\tau,\psi} > 0$. In fact, a summary of these results may be stated as follows.

Chapter 4: (joint work with Romain Gicquaud)

In this chapter, we use a method based on the calculus of variations to show that the smallness of (σ, π) in L^2 is a sufficient condition for existence of solutions to (3):

Theorem 7 (*Small TT-tensor*)**.** *Let data be given on M as specified in (4) and assume that (5) holds. Assume further that $\frac{4(n-1)}{n-2}\Delta + \mathcal{R}_\psi$ is coercive. Then there exists $\epsilon = \epsilon(g, \psi, \tau, V) > 0$ such that if*

$$0 < \int_M \left(|\sigma|^2 + \pi^2\right) dv < \epsilon,$$

the system (3) *has (at least) one solution.*

It is also worth noting that another way to obtain the far from CMC-result in the vacuum case, adding a parameter t in some neighborhood of 0, has been recently presented by Gicquaud–Ngo [Gicquaud et Ngô, 2014] using the implicit function theorem. In Chapter 4, we will also follow this technique that gives another viewpoint on Theorem 7.

Theorem 8 (Another viewpoint on Theorem 7)**.** *Let data be given on M as specified in (4) and assume that (5) holds. Assume further that $\frac{4(n-1)}{n-2}\Delta + \mathcal{R}_\psi$ is coercive. Then there exists $t^* = t^*(g, \tau, \psi, V, \sigma, \pi) > 0$ such that for all $t \in (0, t^*)$, the system (3) associated to*

$(g, \tau, V, \psi, t\sigma, t\pi)$ *has a solution* (φ_t, W_t). *Moreover,* $t^{-\frac{2}{N+2}}\varphi_t$ *is uniformly bounded for all* $t \in (0, t^*)$.

Theorem 8 will be directly used for getting the first solution to (3) with $\mathcal{B}_{\tau,\psi} > 0$ in an effort to show a nonuniqueness result for solutions to the (actual) system (3), which we will present in the next chapter.

Chapter 5:

This chapter is a combination of ideas in the previous chapters to obtain existence and nonuniqueness results for solutions to (3). For existence results, we are interested in giving another proof of Theorem 7 by using the half-continuity method introduced in Chapter 2. Next we will show a nonuniqueness result for solutions to (3) as $\mathcal{B}_{\tau,\psi} > 0$, which is stated as follows.

Theorem 9 (*Nonuniqueness of solutions*)**.** *Let* (M, g) *be a closed locally conformally flat Riemannian manifold of dimension n, with* $3 \leqslant n \leqslant 5$*. Assume that the seed data* $(V, \tau, \psi, \pi, \sigma)$ *given on* M *are smooth and* (M, g) *has no conformal Killing vector field. Assume further that* $\mathcal{B}_{\tau,\psi} > 0$*, and that* $\frac{4(n-1)}{n-2}\Delta + \mathcal{R}_\psi$ *is coercive. If* $\pi \not\equiv 0$*, then there exists a sequence* $\{\epsilon_i\}$ *converging to* 0 *s.t. the system (3) associated to* $(g, \tau, \psi, V, \epsilon_i\sigma, \epsilon_i\pi)$ *has at least two solutions.*

The idea to prove this theorem is that for all $\epsilon > 0$ small enough, by Theorem 8 the system (3) associated to $(g, \tau, \psi, \epsilon\sigma, \epsilon\tau)$ has a solution $(\varphi_\epsilon, W_\epsilon)$ s.t. $\epsilon^{-\frac{2}{N+2}}\varphi_\epsilon$ is uniformly bounded. Thus, it is sufficient to find a solution (φ_i, W_i) to (3) associated to $(g, \tau, \psi, V, \epsilon_i\sigma, \epsilon_i\pi)$ satisfying $\epsilon_i^{-\frac{2}{N+2}}\|\varphi_i\|_{L^\infty} \to \infty$. This will be solved by arguments similar to Chapter 3.

Another issue also arising from general relativity is *the positive mass theorem*. A preamble to this problem may be expressed as follows. The mass of an asymptotically Euclidean manifold (M, g) is an invariant at infinity that appeared first in the context of general relativity as a measure of the total energy of the gravitational field (see [Arnowitt *et al.*, 1960]). In the 80's, R. Schoen showed that the mass naturally appears in conformal geometry and in particular in the solution to the Yamabe problem (see [Lee et Parker, 1987]). The most important result about the mass is the positive mass theorem which states that if (M, g) has non-negative scalar curvature then the mass is non-negative and is zero iff (M, g) is isometric to $\mathbb{R}^n$. This theorem is known to be true when the manifold has dimension less than 8 (see R. Schoen–S.T. Yau [Schoen et Yau, 1979]) or when the manifold is spin (see E. Witten [Witten, 1981], R. Bartnik [Bartnik, 1986]). However, the non-spin case remains completely open in higher dimension. Asymptotically hyperbolic manifolds; i.e., manifolds whose model at infinity is the hyperbolic space, are also of great physical importance and the associated mass was introduced by X. Wang [Wang, 2001] and P. Chruściel–M. Herzlich [Chruściel et Herzlich, 2003]. The corresponding positive mass theorem is known to hold in the spin case but, as for asymptotically Euclidean manifolds, the non-spin case is still open; see however [Andersson *et al.*, 2008]. Recently, E. Humbert–A. Hermann have showed that if the positive mass theorem is true on a closed simply connected non-spin manifold of dimension $n \geqslant 5$, then so it is on all closed manifolds of the same dimension (see [Humbert et Herman, 2014, Theorem 8.5]). This provides a significant reduction: to show the positive mass theorem for closed manifolds of dimension $n \geqslant 5$, it suffices to study a single closed simply connected non-spin manifold of the same dimension. It is then natural to ask if a similar result exists for asymptotically hyperbolic manifolds. In the last chapter we are interested in this question.

Chapter 6:

We consider the positive mass theorem for $C^{2,\alpha}_{\tau}-$asymptotically hyperbolic manifolds of dimension $n \geqslant 5$. The precise definition of such a manifold together with the corresponding statement of the positive mass theorem will be given in Section 6.2. In this context, we will give an answer to the question raised above. We will only partially address the rigidity case of the positive mass theorem so we concentrate mostly on what we call the weak positive mass theorem (weak PMT), namely the fact that the mass vector is future pointing timelike or lightlike. The way that we obtain this answer is similar to the approach of Humbert-Hermann to proving [Humbert et Herman, 2014, Theorem 8.5] in the case of compact Riemannian manifolds. However, the difficulty is here that the technique based on the Green functions of certain second order elliptic operator used in [Humbert et Herman, 2014] cannot be applied for asymptotically hyperbolic manifolds. The result we obtain is the following.

Theorem 10. *Let (N^n, g_0) be a $C^{2,\alpha}_{\tau}-$asymptotically hyperbolic manifold of dimension $n \geqslant 3$ w.r.t. a given diffeomorphism Φ from the exterior of a compact $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$, for some $(\alpha, \delta) \in (0, 1) \times (n/2, n)$. Assume that M^n is obtained from N by a surgery of codimension $q \geqslant 3$. If g_0 does not satisfy the weak PMT, then there exists an asymptotically hyperbolic metric g on M such that $Scal^g \geqslant -n(n-1)$ on M and g does not satisfy the weak PMT.*

Now, by the arguments analogous to [Humbert et Herman, 2014, Theorem 8.5], the answer to our question is givens as follows.

Theorem 11. *If the weak PMT is true on a simply connected non-spin $C^{2,\alpha}_{\tau}-$asymptotically hyperbolic manifold (M, Φ) of dimension $n \geqslant 5$, with $(\alpha, \tau) \in (0, 1) \times (n/2, n)$, then so is it on all asymptotically hyperbolic manifolds of the same dimension.*

To end this chapter, we also give a discussion on the rigidity statement of the positive mass theorem for asymptotically hyperbolic data, which gives us a more interesting view of the relation between the weak positive mass theorem and the strong one.

Theorem 12. *Let M be an open manifold and Φ be a diffeomorphism from the exterior of a compact $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$. Assume that the positive mass theorem for asymptotically Euclidean manifolds is true. Assume further that M satisfies the weak PMT. If there exists a $C^{4,\alpha}_{\tau}-$asymptotically hyperbolic metric g on M whose mass vanishes, then g is isometric to the hyperbolic metric.*

Chapter 1

The Lichnerowicz Equation

1.1 Introduction

The Lichnerowicz equation is the first equation of the system (3). Particularly, it is the special case of (3) when τ is constant (the CMC case). Therefore, the study of the Lichnerowicz equation plays an important role in addressing the system (3). While a complete description of the solutions of the Lichnerowicz equation in the vacuum case was achieved by Isenberg [Isenberg, 1995], the one with a scalar field remains an open problem in general.

In this chapter, we review some standard facts on the Lichnerowicz equation on a compact n−manifold M:

$$\frac{4(n-1)}{n-2}\Delta\varphi + \mathcal{R}_\psi\varphi = \mathcal{B}_{\tau,\psi}\varphi^{N-1} + \left(|\sigma + LW|^2 + \pi^2\right)\varphi^{-N-1}. \tag{1.1}$$

We remind the reader that $\mathcal{R}_\psi = \mathrm{Scal}^g - |d\psi|_g^2$ and $\mathcal{B}_{\tau,\psi} = -\frac{n-1}{n}\tau^2 + 2V(\psi)$. Because treatment of (1.1) depends on the sign of $\mathcal{B}_{\tau,\psi}$, we divide our discussion into two cases.

1.2 The Vacuum Case

In the vacuum case; i.e., $\psi \equiv \pi \equiv 0$ and $V \equiv 0$, the Lichnerowicz equation (1.1) can be rewritten as

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{w^2}{\varphi^{N+1}}, \tag{1.2}$$

with $R = \mathrm{Scal}^g$ and $w = |\sigma + LW|$. Note that τ is here a function. Given a function w and $p > n$, we say that $\varphi_+ \in W^{2,p}_+$ is a *supersolution* to (1.2) if

$$\frac{4(n-1)}{n-2}\Delta\varphi_+ + R\varphi_+ + \frac{n-1}{n}\tau^2\varphi_+^{N-1} \geqslant \frac{w^2}{\varphi_+^{N+1}}.$$

A *subsolution* is defined similarly with the reverse inequality.

Proposition 1.2.1 (see [Isenberg, 1995], [Maxwell, 2005]). *Assume $g \in W^{2,p}$ and $w, \tau \in L^{2p}$ for some $p > n$. If $\varphi_-, \varphi_+ \in W^{2,p}_+$ are a subsolution and a supersolution respectively to (1.2)* *associated with a fixed w such that $\varphi_- \leqslant \varphi_+$, then there exists a solution $\varphi \in W^{2,p}_+$ to (1.2) such that $\varphi_- \leqslant \varphi \leqslant \varphi_+$.*

The next lemma plays an important role in the study on (1.2). It is called the conformal covariance of the Lichnerowicz equation.

Lemma 1.2.2 (see [Isenberg, 1995], [Maxwell, 2009])**.** *Assume $g \in W^{2,p}$ and $w, \tau \in L^{2p}$ for some $p > n$. Assume also that $\theta \in W^{2,p}_+$. Define*

$$\hat{g} = \theta^{N-2} g, \quad \hat{w} = \theta^{-N} w, \quad \hat{\tau} = \tau.$$

Then φ is a supersolution (resp. subsolution) to (1.2) if and only if $\hat{\varphi} = \theta^{-1}\varphi$ is a supersolution (resp. subsolution) to the conformally transformed equation

$$\frac{4(n-1)}{n-2}\Delta_{\hat{g}}\hat{\varphi} + R_{\hat{g}}\hat{\varphi} + \frac{n-1}{n}\hat{\tau}^2\hat{\varphi}^{N-1} = \frac{\hat{w}^2}{\hat{\varphi}^{N+1}}. \tag{1.3}$$

In particular, φ is a solution to (1.2) if and only if $\hat{\varphi}$ is a solution to (1.3).

Proof. Let $g' = \varphi^{N-2} g$, and let $R_{g'}$ be its scalar curvature. Then it is well known that

$$R_{g'} = \varphi^{-N+1}\left(\frac{4(n-1)}{n-2}\Delta_g\varphi + R_g\varphi\right).$$

But $g' = \left(\theta^{-1}\varphi\right)^{N-2}\hat{g}$, then

$$R_{g'} = \theta^{N-1}\varphi^{-N+1}\left(\frac{4(n-1)}{n-2}\Delta_{\hat{g}}\left(\theta^{-1}\varphi\right) + R_{\hat{g}}\left(\theta^{-1}\varphi\right)\right).$$

Hence

$$\begin{aligned}
&\frac{4(n-1)}{n-2}\Delta_{\hat{g}}\hat{\varphi} + R_{\hat{g}}\hat{\varphi} + \frac{n-1}{n}\hat{\tau}^2\hat{\varphi}^{N-1} - \hat{w}^2\hat{\varphi}^{-N-1}\\
&\qquad = \frac{4(n-1)}{n-2}\Delta_{\hat{g}}\left(\theta^{-1}\varphi\right) + R_{\hat{g}}\left(\theta^{-1}\varphi\right) + \frac{n-1}{n}\hat{\tau}^2\left(\theta^{-1}\varphi\right)^{N-1} - \hat{w}^2\left(\theta^{-1}\varphi\right)^{-N-1}\\
&\qquad = \theta^{-N+1}\left(\frac{4(n-1)}{n-2}\Delta_g\varphi + R_g\varphi\right) + \theta^{-N+1}\frac{n-1}{n}\tau^2\varphi^{N-1} - \theta^{-N+1}w^2\varphi^{-N-1}\\
&\qquad = \theta^{-N+1}\left(\frac{4(n-1)}{n-2}\Delta_g\varphi + R_g\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} - w^2\varphi^{-N-1}\right).
\end{aligned}$$

The result now follows by noticing that $\theta^{-N+1} > 0$ everywhere. □

We are now ready to prove the main theorem in this section.

Theorem 1.2.3. *(see [Maxwell, 2005]) Assume $w, \tau \in L^{2p}$ and $g \in W^{2,p}$ for some $p > n$. Then there exists a positive solution $\varphi \in W^{2,p}_+$ to (1.2) if and only if one of the following assertions is true.*

1. $\mathcal{Y}_g > 0$ and $w \not\equiv 0$,

2. *$\mathcal{Y}_g = 0$ and $w \not\equiv 0$, $\tau \not\equiv 0$,*

3. *$\mathcal{Y}_g < 0$ and there exists $\hat{g}$ in the conformal class of g such that $R_{\hat{g}} = -\frac{n-1}{n}\tau^2$,*

4. *$\mathcal{Y}_g = 0$ and $w \equiv 0$, $\tau \equiv 0$.*

In Cases 1 − 3 the solution is unique. In Case 4 any two solutions are related by a scaling by a positive constant multiple. Moreover, Case 3 holds if $\mathcal{Y}_g < 0$ and the set of all zero-points of τ has zero Lebesgue measure (see [Aubin, 1998, Theorem 6.12]).

Proof. • **Existence of solution:** By Lemma 1.2.2 without loss of generality we may assume that $R > 0$ or $R \equiv 0$ or $R < 0$ depending on the sign of the Yamabe invariant. Since the situation 4) is trivial, we may assume that at least one of w and τ is nonzero and then divide our arguments into two possibilities.

Cases 1 and 2: Since $\left(R + \frac{n-1}{n}\tau^2\right)$ is non-negative and not identically zero, there exists a solution $\varphi_1 \in W^{2,p}$ to

$$\frac{4(n-1)}{n-2}\Delta\varphi_1 + \left(R + \frac{n-1}{n}\tau^2\right)\varphi_1 = w^2.$$

Since $w \not\equiv 0$, if follows from the maximum principle (see [Maxwell, 2005, Lemma 2.9 and Proposition 2.10]) that $\varphi_1 > 0$. For any given $\lambda \in \mathbb{N}^*$, we have that

$$\begin{aligned}\frac{4(n-1)}{n-2}&\Delta(\lambda\varphi_1) + R(\lambda\varphi_1) + \frac{n-1}{n}\tau^2(\lambda\varphi_1)^{N-1} - w^2(\lambda\varphi_1)^{-N-1}\\ &= \frac{n-1}{n}\tau^2\left((\lambda\varphi_1)^{N-1} - \lambda\varphi_1\right) + w^2\left(\lambda - (\lambda\varphi_1)^{-N-1}\right).\end{aligned}$$

Therefore, it is easy to see that $\lambda\varphi_1$ is a supersolution to (1.2), provided λ is large. Similarly, we also have that $\lambda^{-1}\varphi_1$ is a subsolution to (1.2) for large λ. It then follows from Proposition 1.2.1 that (1.2) admits a solution φ satisfying

$$0 < \lambda^{-1}\varphi_1 \leqslant \varphi \leqslant \lambda\varphi_1.$$

To prove the opposite direction, we need to show that if there is a solution with $\mathcal{Y}_g \geqslant 0$, and if at least one of $\tau \equiv 0$ or $w \equiv 0$, then either $\mathcal{Y}_g > 0$ and $w \not\equiv 0$ or $\mathcal{Y}_g = 0$ and w and τ both vanish identically. In fact, if a positive solution φ to (1.2) exists, we can set $\hat{g} = \varphi^{N-2}g$ and $\hat{w} = \varphi^{-N}w$ to obtain

$$R_{\hat{g}} = \hat{w}^2 - \frac{n-1}{n}\tau^2. \tag{1.4}$$

If $w \equiv 0$ and therefore $\hat{w} \equiv 0$, then we obtain from (1.4) that $\mathcal{Y}_{\hat{g}} \leqslant 0$. Since $\mathcal{Y}_g \geqslant 0$ we have $\mathcal{Y}_g = \mathcal{Y}_{\hat{g}} = 0$ and $\tau^2 \equiv 0$. On the other hand, if $\tau^2 \equiv 0$ and $w \not\equiv 0$, then we obtain $\mathcal{Y}_g > 0$ (see [Maxwell, 2005, Corollary 3.4]).

Case 3: In the situation $w \equiv 0$ there is nothing to prove. We can then assume that $w \not\equiv 0$. First suppose that there exists a conformal factor $\varphi_0 > 0$ such that $g_0 = \varphi^{N-2}g$ satisfies $R_{g_0} = -\frac{n-1}{n}\tau^2$. Since $\mathcal{Y}_g < 0$ this implies in particular that $\tau \not\equiv 0$. Solving the Lichnerowicz equation by Lemma 1.2.2 reduces to solving

$$\frac{4(n-1)}{n-2}\Delta_{g_0}\varphi - \frac{n-1}{n}\tau^2\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + w_0^2\varphi^{-N-1},$$

with $w_0 = \varphi_0^{-N} w$. Let $\varphi_1 \in W^{2,p}$ be the unique solution to

$$\frac{4(n-1)}{n-2}\Delta_{g_0}\varphi + \frac{n-1}{n}\tau^2\varphi = w_0^2.$$

Since $\tau \not\equiv 0$ and since $w_0 \not\equiv 0$, the solution exists and is positive. Similarly to arguments above, we obtain that $\lambda\varphi_1$ (resp. $\lambda^{-1}\varphi_1$) is a supersolution (resp. subsolution) to (1.2), and then by Proposition 1.2.1 there exists a solution φ to (1.2) s.t.

$$0 < \lambda^{-1}\varphi_1 \leqslant \varphi \leqslant \lambda\varphi_1.$$

For the converse, assume that $\varphi > 0$ is a solution to (1.2). We wish to find a solution to the equation

$$\frac{4(n-1)}{n-2}\Delta v + Rv = -\frac{n-1}{n}\tau^2 v^{N-1}. \tag{1.5}$$

Since $\mathcal{Y}_g < 0$ (i.e. $R < 0$ by Lemma 1.2.2), $R\epsilon + \frac{n-1}{n}\tau^2\epsilon^{N-1} < 0$ for small $\epsilon > 0$, and hence $v_0 = \epsilon$ is a subsolution to (1.5). On the other hand, notice that φ is also a supersolution, thus for some given small $\epsilon \leqslant \min\varphi$ we have by Proposition 1.2.1 that (1.5) has a solution v s.t. $0 < v_0 \leqslant v \leqslant \varphi$ as claimed.

• **Uniqueness of solution:** Assume that $\varphi_1, \varphi_2 > 0$ are solutions to (1.2). Let $\hat{g} = \varphi_1^{N-2} g$, $\hat{w} = \varphi_1^{-N} w$ and $\varphi = \varphi_2/\varphi_1$. Then φ solves

$$\frac{4(n-1)}{n-2}\Delta_{\hat{g}}\varphi + \hat{w}^2\varphi - \frac{n-1}{n}\tau^2\varphi = \hat{w}^2\varphi^{-N-1} - \frac{n-1}{n}\tau^2\varphi^{N-1}.$$

Hence $\varphi - 1$ satisfies

$$\frac{4(n-1)}{n-2}\Delta(\varphi - 1) + \left(\hat{w}^2 - \frac{n-1}{n}\tau^2\right)(\varphi - 1) = \hat{w}^2\left(\varphi^{-N-1} - 1\right) - \frac{n-1}{n}\tau^2\left(\varphi^{N-1} - 1\right). \tag{1.6}$$

Multiplying (1.6) by $(\varphi - 1)^+$ and integrating over M we have

$$\int_{\varphi>1} \frac{4(n-1)}{n-2}\langle\nabla(\varphi-1), \nabla(\varphi-1)\rangle_{\hat{g}} dv = \int_{\varphi>1}\left[\hat{w}^2\left(\varphi^{-N-1} - \varphi\right)(\varphi - 1) + \frac{n-1}{n}\tau^2\left(\varphi - \varphi^{N-1}\right)(\varphi - 1)\right] dv.$$

If $\varphi \geqslant 1$ then $\left(\varphi^{-N-1} - \varphi\right)(\varphi - 1) \leqslant 0$ and $\left(\varphi - \varphi^{N-1}\right)(\varphi - 1) \leqslant 0$. So the integral on the right hand side of the inequality above must be non-positive. Since the integral on the left hand side is non-negative, we conclude

$$\begin{aligned} \int_{\varphi>1} \frac{4(n-1)}{n-2}\langle\nabla(\varphi-1), \nabla(\varphi-1)\rangle_{\hat{g}} dv &= 0 \\ \int_{\varphi>1}\left[\hat{w}^2\left(\varphi^{-N-1} - \varphi\right)(\varphi - 1) + \frac{n-1}{n}\tau^2\left(\varphi - \varphi^{N-1}\right)(\varphi - 1)\right] dv &= 0. \end{aligned} \tag{1.7}$$

A similar argument using $(\varphi - 1)^-$ as a test function shows

$$\begin{aligned} \int_{\varphi<1} \frac{4(n-1)}{n-2}\langle\nabla(\varphi-1), \nabla(\varphi-1)\rangle_{\hat{g}} dv &= 0 \\ \int_{\varphi<1}\left[\hat{w}^2\left(\varphi^{-N-1} - \varphi\right)(\varphi - 1) + \frac{n-1}{n}\tau^2\left(\varphi - \varphi^{N}\right)(\varphi - 1)\right] dv &= 0. \end{aligned} \tag{1.8}$$

From the continuity of $\varphi - 1$ and the first equations of (1.7) and (1.8) we conclude φ is constant and therefore φ_1 is a constant multiple of φ_2. Suppose that $\varphi_1 \not\equiv \varphi_2$, so $\varphi \not\equiv 1$. Then from the remaining equations of (1.7) and (1.8) we obtain

$$\int_M \left[\hat{w}^2 \left(\varphi^{-N-1} - \varphi \right) (\varphi - 1) + \frac{n-1}{n} \tau^2 \left(\varphi - \varphi^{N-1} \right) (\varphi - 1) \right] dv = 0$$

and therefore $\hat{w} \equiv 0$, $\tau \equiv 0$ and $\mathcal{Y}_g = 0$. The proof is completed. □

From the techniques in [Gicquaud et Ngô, 2014], we also get the following remark, which allows us to assume that R has a constant sign without loss generality.

Remark 1.2.4. *Theorem 1.2.3 guarantees that given any $w \in L^{2p} \setminus \{0\}$, there exists a unique solution $\varphi \in W^{2,p}_+$ to (1.2). In addition, by direct calculation, we compute for any $k \geqslant N$*

$$\int_M \hat{\varphi}^k dv_{\hat{g}} = \int_M \theta^{N-k} u^k dv_g \quad \textit{and} \quad \int_M \hat{w}^k dv_{\hat{g}} = \int_M \theta^{N(1-k)} w^k dv_g,$$

where $(\hat{g}, \hat{\varphi}, \hat{w})$ is given as in Lemma 1.2.2. It follows that

$$(\max \theta)^{\frac{N-k}{k}} \|\varphi\|_{L^k_g} \leqslant \|\hat{\varphi}\|_{L^k_{\hat{g}}} \leqslant (\min \theta)^{\frac{N-k}{k}} \|\varphi\|_{L^k_g}$$

and

$$(\max \theta)^{\frac{N(1-k)}{k}} \|w\|_{L^k_g} \leqslant \|\hat{w}\|_{L^k_{\hat{g}}} \leqslant (\min \theta)^{\frac{N(1-k)}{k}} \|w\|_{L^k_g}.$$

Without loss of generality, we can assume moreover that $R > 0$ or $R \equiv 0$ or $R = -\frac{n-1}{n}\tau^2$ depending on the sign of $\mathcal{Y}_g$ (in the case $\mathcal{Y}_g < 0$, we refer to Case 3 of Lemma 1.2.3). Under this assumption, it is also helpful to keep in mind that the term $R\varphi^{k+1} + \frac{n-1}{n}\tau^2\varphi^{k+N-1}$ is uniformly bounded from below for all positive functions $\varphi \in L^\infty$ and all $k \geqslant 0$. In fact, if $R \geqslant 0$, it is obvious that $R\varphi^{k+1} + \frac{n-1}{n}\tau^2\varphi^{k+N-1} \geqslant 0$. If $R = -\frac{n-1}{n}\tau^2$, then $\frac{n-1}{n}\tau^2\varphi^{k+1}\left(\varphi^{N-2} - 1\right) \geqslant -\frac{n-1}{n}(\max|\tau|)^2$, which is our claim.

The following lemma will be used throughout the thesis.

Lemma 1.2.5. *Assume that ϕ, φ are a supersolution (resp. subsolution) and a positive solution respectively to (1.2) associated with a fixed w. Then*

$$\phi \geqslant \varphi \ (\textit{resp.} \leqslant).$$

In particular, assume that φ_0 (resp. φ_1) is a positive solution to (1.2) associated to $w = w_0$ (resp. w_1). Assume moreover $w_0 \leqslant w_1$. Then $\varphi_0 \leqslant \varphi_1$.

We give a simple proof of this fact based on Theorem 1.2.3 (even if the proof of Theorem 1.2.3 requires the maximum principle). Another proof, independent of Theorem 1.2.3, can be found in [Dahl *et al.*, 2012].

Proof. We will prove the supersolution case. The remaining cases are similar. Assume that ϕ, φ are a supersolution and a positive solution respectively of (1.2) associated to a fixed w. Since φ is a solution, φ is also a subsolution, and hence, as easily checked so is $t\varphi$ for all constant $t \in (0, 1]$. Since $\min \phi > 0$, we now take t small enough s.t. $t\varphi \leqslant v$. By Proposition 1.2.1, we then conclude that there exists a solution $\varphi' \in W^{2,p}$ to (1.2) satisfying $t\varphi \leqslant \varphi' \leqslant \phi$. On the other hand, by uniqueness of positive solutions to (1.2) given by Theorem 1.2.3, we obtain that $\varphi = \varphi'$, and hence get the desired conclusion. □

1.3 The Scalar Field Case

In the scalar field case; i.e., for a general (ψ, V, π), for shortened notation, we may rewrite the Lichnerowicz equation as follows:

$$\Delta\varphi + h\varphi = B\varphi^{N-1} + A\varphi^{-N-1}, \tag{1.9}$$

where $h = \frac{n-2}{4(n-1)}\mathcal{R}_\psi$, $B = \frac{n-2}{4(n-1)}\mathcal{B}_{\tau,\psi}$ and $A = \frac{n-2}{4(n-1)}\left(|\sigma + LW|^2 + \pi^2\right)$. Unlike the vacuum case, where we know exactly for which seed data the Lichnerowicz equation admits solutions, there remain many unanswered problems for the scalar field case. As evidence for the complications of this case, we will review some recent results which show that (1.9) may have no solution, exactly one or at least two solutions under certain conditions. With the notation $h^+ = \max\{h, 0\}$, we first consider the possibility of nonexistence.

Proposition 1.3.1 (Hebey–Pacard–Pollack [Hebey *et al.*, 2008])**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. Assume that h, B and A are smooth functions on M. If $A, B > 0$ and*

$$\max_M \left(\frac{(h^+)^n}{AB^{n-1}}\right) < \frac{n^n}{(n-1)^{n-1}} \tag{1.10}$$

on M, then the Lichnerowicz equation (1.9) has no smooth positive solution.

Proof. We argue by contradiction. Assume that φ is a smooth solution to (1.9). Let m_0 be any minimum point of φ. Since $\Delta\varphi(m_0) \leqslant 0$, we have from (1.9) that

$$h\varphi(m_0) \geqslant \left\{B\varphi^{N-1} + A\varphi^{-N-1}\right\}(m_0).$$

Since both A and B are positive, it follows from the inequality above that

$$h \geqslant \left\{B\varphi^{N-2} + A\varphi^{-N-2}\right\}(m_0),$$

and then by standard calculation we obtain that

$$\max_M \left(\frac{(h^+)^n}{AB^{n-1}}\right) \geqslant \frac{n^n}{(n-1)^{n-1}},$$

which is a contradiction with (1.10). The proof is completed. □

We now obtain another nonexistence result involving the Lebesgue norms of the functions h, B and A.

Theorem 1.3.2 (Hebey–Pacard–Pollack [Hebey *et al.*, 2008])**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. Assume that h, B and A are smooth functions on M. If $B > 0$ on M, and*

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv > \int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv, \tag{1.11}$$

then the Lichnerowicz equation (1.9) has no smooth positive solution.

Proof. Let φ be a smooth positive solution to (1.9). Integrating (1.1) over M we obtain that

$$\int_M B\varphi^{N-1}dv + \int_M \frac{A}{\varphi^{N+1}}dv = \int_M h\varphi dv. \tag{1.12}$$

By Hölder's inequality,

$$\int_M h\varphi dv \leqslant \left(\int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv\right)^{\frac{4}{n+2}} \left(\int_M B\varphi^{N-1}dv\right)^{\frac{n-2}{n+2}}.$$

Again by using Hölder's inequality,

$$\int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv \leqslant \left(\int_M B\varphi^{N-1}dv\right)^{\frac{3n-2}{4n}} \left(\int_M \frac{A}{\varphi^{N+1}}dv\right)^{\frac{n+2}{4n}}.$$

Collecting these inequalities and using (1.12), we get

$$X + \left(\int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv\right)^{\frac{4n}{n+2}} X^{1-n} \leqslant \left(\int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv\right)^{\frac{4}{n+2}}, \tag{1.13}$$

where we have set

$$X = \left(\int_M B\varphi^{N-1}dv\right)^{\frac{4}{n+2}}.$$

The study of the minimal value of the function X which appears on the left hand side of (1.13) implies that

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)\left(\int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv\right)^{\frac{4n}{n+2}} \leqslant \left(\int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv\right)^{\frac{4n}{n+2}}.$$

This completes the proof of the theorem. □

We next recall existence results for (1.9). For the proofs of theorems below we refer the reader to [Hebey *et al.*, 2008] and [Ngo et Xu, 2012]. Before going further, we introduce the following necessary definitions. Assume that h is chosen s.t. $\Delta + h$ is coercive. Then we may define

$$\|\varphi\|_{H_h^1} = \left(\int_M \left(|\nabla\varphi|^2 + h\varphi^2\right) dv\right)^{\frac{1}{2}}. \tag{1.14}$$

We also denote by $S_\psi = S(M, g, \psi) > 0$, the Sobolev constant defined as the smallest constant $S_\psi > 0$ s.t.

$$\int_M |\varphi|^N dv \leqslant S_\psi \left(\int_M \left(|\nabla\varphi|^2 + h\varphi^2\right) dv\right)^{\frac{N}{2}} \tag{1.15}$$

for all $\varphi \in H^1(M)$.

Theorem 1.3.3. *(see [Hebey et al., 2008, Theorem 3.1]) Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. Assume that h, A, B are smooth functions, $\Delta + h$ is coercive, $A > 0$ and $\max B > 0$. There exists a constant $C = C(n) > 0$ such that if*

$$\|u\|_{H_h^1}^N \int_M \frac{A}{u^N}dv \leqslant \frac{C}{\left(S_\psi \max|B|\right)^{n-1}} \tag{1.16}$$

and

$$\int_M Bu^N dv > 0$$

for some smooth positive function $u > 0$ *in* M*, where* $\|.\|_{H_h^1}$ *is as in (1.14) and* S_ψ *is as in (1.15), then the Lichnerowicz equation (1.9) has at least one smooth positive solution.*

As a remark, it can be noted that when $\int_M Bdv > 0$, then we can take φ to be constant in (1.16). In particular, this existence result has the following corollary.

Corollary 1.3.4. *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geqslant 3$*. Assume that* $\Delta + h$ *is coercive,* $A > 0$*,* $\max B > 0$ *and* $\int_M Bdv > 0$*. There exists a constant* $C = C(n, g, \psi) > 0$ *such that if*

$$\left(\max_M |B|\right)^{n-1} \int_M Adv \leqslant C,$$

then (1.9) has a smooth positive solution.

When $A > 0$ and $B > 0$, we can also take $\varphi = A^{\frac{n-2}{4n}}$ in (1.16), so Theorem 1.3.3 has the following corollary.

Corollary 1.3.5. *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geqslant 3$*. Assume that* $\Delta + h$ *is coercive,* $A > 0$ *and* $B > 0$*. There exists a constant* $C = C(n, g, \psi) > 0$ *such that if*

$$(\max B)^{n-1} \left\|A^{\frac{n-2}{4n}}\right\|_{H^1}^N \int_M A^{\frac{1}{2}} dv \leqslant C,$$

then (1.9) has a smooth positive solution.

The next result provides nonuniqueness of solutions to (1.9). For a given $f \in L^\infty$, we define

$$\mathcal{Y}_B = \begin{cases} \inf_{\varphi \in \mathcal{K}} \frac{\int_M |\nabla\varphi|^2 dv}{\int_M |\varphi|^2 dv} & \text{if } \mathcal{K} \neq \emptyset \\ +\infty & \mathcal{K} = \emptyset, \end{cases} \tag{1.17}$$

where

$$\mathcal{K} = \left\{\varphi \in H^1(M) : \ \varphi \geqslant 0, \ \varphi \not\equiv 0, \ \int_M |B^-|\varphi dv = 0\right\}.$$

We have the following theorem.

Theorem 1.3.6 (Nonuniqueness of solutions, see [Ngo et Xu, 2012])**.** *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geqslant 3$*. Assume that* B *is a smooth function s.t.* $\int_M Bdv < 0$*,* $\sup B > 0$ *and* $|h| < \mathcal{Y}_B$ *where* $\mathcal{Y}_B$ *is given in (1.17). Suppose that*

$$\int_M Adv < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |B^-|\, dv}\right)^n \int_M \left|B^-\right| dv.$$

Then there exists a number $C > 0$ *such that if*

$$\frac{\sup B}{\int_M |B^-|\, dv} < C,$$

the Lichnerowicz equation (1.9) has at least two solutions.

To end this chapter we would like to review a result proven by Premoselli [Premoselli, 2015], which gives us a more general view of the set of all solutions to (1.9). A shortcoming of this result is, however, that it is only proven on compact Riemannian manifolds of dimension $3 \leqslant n \leqslant 5$; the remaining cases are still unknown.

Theorem 1.3.7 (see Premoselli [Premoselli, 2014])**.** *Let (M, g) be a compact Riemannian manifold of dimension n, with $3 \leqslant n \leqslant 5$, and assume that A, B and h are smooth functions on M. Assume further that $\Delta + h$ is coercive, $A \not\equiv 0$ and $\max B > 0$. Then there exist $0 < \theta_1 \leqslant \theta_2 \leqslant +\infty$ such that Equation (1.9) associated to $(h, B, \theta A)$ has:*

- *at least two solutions if $\theta < \theta_1$,*
- *at least one solution if $\theta_1 \leqslant \theta < \theta_2$,*
- *no solutions for $\theta > \theta_2$.*

Moreover if $B > 0$ on M, then $\theta_1 = \theta_2 < +\infty$, and hence there exists $\theta^* \in (0, +\infty)$ such that the associated (1.9) has at least two solutions if $\theta < \theta^*$, exactly one solution if $\theta = \theta^*$, and no solution if $\theta > \theta^*$.

Chapter 2

Applications of Fixed Point Theorems to the Vacuum Einstein Constraint Equations with Non-Constant Mean Curvature

2.1 Introduction

We may say that until now the two most striking results to the vacuum conformal constraint equations (3) in the far-from-CMC regime have been the following:

Theorem 2.1.1 (Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009] and Maxwell [Maxwell, 2009])**.** *Let data be given on M as specified in (4) associated to the vacuum case and assume that conditions (5) hold. Assume further that the Yamabe invariant $\mathcal{Y}_g > 0$. Then there exists $\epsilon(g, \tau) > 0$ such that if*

$$0 < \|\sigma\|_{L^\infty} < \epsilon, \tag{2.1}$$

the system (3) *has (at least) one solution.*

Theorem 2.1.2 (Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012])**.** *Let data be given on M as specified in (4) associated to the vacuum case and assume that conditions (5) hold. Furthermore, assume that $\tau > 0$. Then at least one of the following assertions is true*

- *The constraint equations (3) admit a solution (φ, W) with $\varphi > 0$. Furthermore, the set of solutions $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$ is compact.*
- *There exists a nontrivial solution $W \in W^{2,p}$ to the limit equation*

$$-\frac{1}{2}L^*LW = \alpha\sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{\tau}, \tag{2.2}$$

for some $\alpha \in (0, 1]$.

The main tool these authors use for obtaining the theorems above is the Schauder fixed point theorem.

In this chapter, we will introduce two other fixed point theorems which turn out to be more efficient than the Schauder fixed point theorem for solving (3): Leray-Schauder's fixed point theorem and a generalization of Schauder's fixed point theorem for half-continuous maps. As an early indication of this statement, our main aim in this chapter is to use these methods for showing a stronger result and a simpler proof respectively to the two theorems above. Furthermore, we also give unifying viewpoint of them.

The outline of this chapter is as follows. In Section 2.2, we will give another proof of Theorem 2.1.2. Next we will deeply discuss this result (see Proposition 2.2.4-2.2.10); for instance, Proposition 2.2.4 and 2.2.5 show that the non-vanishing assumption of τ plays an important role in the proof, and hence the proof fails to extend Theorem 2.2 for a general τ. As a second example, we also obtain in Proposition 2.2.10 a situation where both assertions in Theorem 2.1.2 are true.

In Section 2.3, we first review fixed point theorems for half-continuous maps, which may be a little strange for the reader. Next we will apply this concept to prove L^n-near CMC results as stated by Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012] or Gicquaud–Sakovich [Gicquaud et Sakovich, 2012] for asymptotically hyperbolic manifolds; i.e. the vacuum conformal constraint equations (3) has a solution, provided $d\tau/\tau$ is small enough in L^n. Finally, we will prove two theorems in this section, which show efficiency of the half-continuity method. The first is a sharpening of the estimate (2.1) in Theorem 2.1.1, while the latter shows that the assumption of the existence of global supersolutions used in [Holst *et al.*, 2009] and [Maxwell, 2009] to solve the vacuum system (3) can be weakened: the existence of local supersolutions, whose definition is given in Subsection 2.3.3, is sufficient here. More precisely, we have the following results:

Theorem 2.1.3 (***Small TT-tensor***)**.** *Let data be given on M as specified in (4) associated to the vacuum case and assume that (5) holds. Assume further that the Yamabe invariant $\mathcal{Y}_g > 0$. Then there exists $\epsilon(g, \tau) > 0$ such that if*

$$0 < \int_M |\sigma|^2 dv < \epsilon, \tag{2.3}$$

the system (3) *has (at least) one solution.*

Theorem 2.1.4 (***Local supersolution***)**.** *Let data be given on M as specified in (4) associated to the vacuum case and assume that (5) holds. Assume further that $\theta \in L^\infty_+$ is a local supersolution to (3). Then (3) admits a solution.*

Efficiency of these two methods will also be apparent in the next chapters.

2.2 A New Proof for Theorem 2.1.2

In this section we show how Schaefer's fixed point theorem can be applied to give a simpler proof of the main result in [Dahl *et al.*, 2012]. We first recall its statement (see [Gilbarg et Trudinger, 2001, Theorem 11.6]).

Theorem 2.2.1. *(**Leray-Schauder's fixed point**) Let X be a Banach space and assume that $S : X \times [0,1] \to X$ is a continuous compact mapping, satisfying $S(x,0) = 0$ for all $x \in X$. If the set*

$$K = \{x \in X |\ \exists t \in [0,1] \text{ such that } x = S(x,t)\}$$

is bounded, then $S = S(.,1)$ has a fixed point.

Corollary 2.2.2. *(**Schaefer's fixed point**) Assume that $S : X \to X$ is continuous compact and that the set*

$$K = \{x \in X | \exists t \in [0,1] \text{ such that } x = tS(x)\}$$

is bounded. Then S has a fixed point.

Before going further, it is worth reviewing that Dahl–Gicquaud–Humbert's proof for Theorem 2.1.2 in [Dahl *et al.*, 2012] goes as follows: first, they apply Schauder's fixed point theorem to solve a subcritical system, that is a small perturbation of the system (3) involving subcritical exponents. This provides a sequence (u_ϵ) of subcritical solutions which is expected to converge to a solution to (3) when ϵ tends to 0. A study of the sequence (u_ϵ) shows that this actually happens when the limit equation (2.2) has no non-trivial solution.

In the proof we present here, we show that Shaefer's fixed point theorem can be applied as soon as (2.2) has no non-trivial solution, leading directly to the existence of a solution to (3). This simplifies the proof.

Similarly to [Dahl *et al.*, 2012] and [Maxwell, 2009], first we need to introduce the following map. Throughout this chapter, we define the map $T : L^\infty \to L^\infty$ as follows. Given data on M as specified in (4) and assuming that (5) holds, for each $\varphi \in L^\infty$, there exists a unique $W \in W^{2,p}$ such that

$$-\frac{1}{2} L^* LW = \frac{n-1}{n} \varphi^N d\tau,$$

and there is a unique $\theta \in W^{2,p}_+$ satisfying (see Theorem 1.2.3)

$$\frac{4(n-1)}{n-2} \Delta\theta + R\theta = -\frac{n-1}{n} \tau^2 \theta^{N+1} + |\sigma + LW|^2 \theta^{-N-1}.$$

We define

$$T(\varphi) = \theta.$$

Proposition 2.2.3 (see [Dahl *et al.*, 2012], or [Maxwell, 2009])**.** *T is continuous compact and $T(\varphi) > 0$ for all $\varphi \in L^\infty$.*

We are now ready to give the proof.

Another proof of Theorem 2.1.2. Let T be the continuous compact operator given as above. Set

$$S = \{\varphi \in L^\infty /\ \exists t \in [0,1] : \ \varphi = tT(\varphi)\}.$$

If S is bounded, we get a solution to (3) by Corollary 2.2.2. If S is not bounded, there exists a sequence $(t_i, \varphi_i) \in [0.1] \times L^\infty$ with $\|\varphi_i\|_{L^\infty} \to \infty$ such that

$$\frac{4(n-1)}{n-2}\Delta\theta_i + R\theta_i = -\frac{n-1}{n}\tau^2\theta_i^{N-1} + |\sigma + LW_i|^2\theta_i^{-N-1} \tag{2.4a}$$

$$-\frac{1}{2}L^*LW_i = \frac{n-1}{n}\varphi_i^N d\tau, \tag{2.4b}$$

where $\theta_i = T(\varphi_i)$ and $\varphi_i = t_i\theta_i$. We modify the main idea in [Dahl *et al.*, 2012] to obtain the (non-trivial) solution to the limit equation. We set $\gamma_i = \|\theta_i\|_\infty$ and rescale θ_i, W_i and σ as follows:

$$\widetilde{\theta}_i = \gamma_i^{-1}\theta_i, \quad \widetilde{W}_i = \gamma_i^{-N}W_i, \quad \widetilde{\sigma}_i = \gamma_i^{-N}\sigma.$$

It may be worth noticing that $\gamma_i = \|\theta_i\|_\infty = \frac{1}{t_i}\|\varphi_i\|_\infty \to \infty$ as $i \to \infty$. The system (2.4), with φ_i replaced by $t_i\theta_i$ in the vector equation, can be rewritten as

$$\frac{1}{\gamma_i^{N-2}}\left(\frac{4(n-1)}{n-2}\Delta\widetilde{\theta}_i + R\widetilde{\theta}_i\right) = -\frac{n-1}{n}\tau^2\widetilde{\theta}_i^{N-1} + |\widetilde{\sigma} + L\widetilde{W}_i|^2\widetilde{\theta}_i^{-N-1} \tag{2.5a}$$

$$-\frac{1}{2}L^*L\widetilde{W}_i = \frac{n-1}{n}t_i^N\widetilde{\theta}_i^N d\tau. \tag{2.5b}$$

Since $\|\widetilde{\theta}_i\|_\infty = 1$, we conclude from the vector equation that $\left(\widetilde{W}_i\right)_i$ is bounded in $W^{2,p}$ and then by the Sobolev embedding, (after passing to a subsequence) $\widetilde{W}_i$ converges in the C^1-norm to some $\widetilde{W}_\infty$. We now prove that

$$\widetilde{\theta}_i \to \left(\sqrt{\frac{n}{n-1}}\frac{|L\widetilde{W}_\infty|}{\tau}\right)^{\frac{1}{N}} \quad \text{in } L^\infty. \tag{2.6}$$

Note that if such a statement is proven, passing to the limit in the vector equation, we see that $\widetilde{W}_\infty$ is a solution to the limit equation with (after passing to a subsequence) $\alpha_0 = \lim t_i^N \in [0,1]$. On the other hand, since $\|\widetilde{\theta}_i\|_\infty = 1$ for all i, $\widetilde{W}_\infty \not\equiv 0$ from (2.6) and then by the assumption that (M, g) has no conformal Killing vector field, we obtain that $\alpha_0 \neq 0$ which completes the proof.

For any $\epsilon > 0$, since $\frac{|L\widetilde{W}_\infty|}{\tau} \in C^0$, we can choose $\widetilde{\omega} \in C^2_+$ s.t.

$$\left|\widetilde{\omega} - \left(\sqrt{\frac{n}{n-1}}\frac{|L\widetilde{W}_\infty|}{\tau}\right)^{\frac{1}{N}}\right| < \frac{\epsilon}{2}. \tag{2.7}$$

To show (2.6), it suffices to prove that

$$|\widetilde{\theta}_i - \widetilde{\omega}| \leqslant \frac{\epsilon}{2}$$

for all i large enough. We argue by contradiction. Assume that the previous inequality is not true. We first consider the case when (after passing to a subsequence) there exists a sequence $(m_i) \in M$ s.t.

$$\widetilde{\theta}_i(m_i) > \widetilde{\omega}(m_i) + \frac{\epsilon}{2}. \tag{2.8}$$

By Lemma 1.2.5 and Inequality (2.8), $\widetilde{\omega} + \frac{\epsilon}{2}$ is not a supersolution to the rescaled Lichnerowicz equation. As a consequence, since Δ is here assumed to be the nonnegative Laplace operator, there exists a sequence $(p_i) \in M$ satisfying

$$\frac{1}{\gamma_i^{N-2}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i) + R\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i)\right] + \frac{n-1}{n}\tau^2(p_i)\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{N-1}(p_i)$$
$$< \left|\widetilde{\sigma}_i(p_i) + L\widetilde{W}_i(p_i)\right|^2\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{-N-1}(p_i).$$

Without loss of generality, we can assume that there exists $p_\infty \in M$ such that $p_i \to p_\infty$. Since $\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)$ and τ are positive, the previous inequality can be rewritten as follows

$$\frac{n\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{N+1}(p_i)}{(n-1)\tau^2(p_i)\gamma_i^{N-2}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i) + R\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i)\right] + \left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{2N}(p_i)$$
$$< \frac{n}{n-1}\left|\widetilde{\sigma}_i(p_i) + L\widetilde{W}_i(p_i)\right|^2\tau^{-2}(p_i).$$

Taking $i \to \infty$, due to the facts that $\widetilde{\omega} \in C_+^2$, $\min\tau > 0$, $\gamma_i \to \infty$ and $\widetilde{W}_i \to \widetilde{W}_\infty$ in C^1–norm, we obtain that

$$\frac{n\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{N+1}(p_i)}{(n-1)\tau^2(p_i)\gamma_i^{N-2}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i) + R\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)(p_i)\right] \to 0,$$

$$\left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{2N}(p_i) \to \left(\widetilde{\omega}+\frac{\epsilon}{2}\right)^{2N}(p_\infty)$$

and

$$\frac{n}{n-1}\left|\widetilde{\sigma}_i(p_i) + L\widetilde{W}_i(p_i)\right|^2\tau^{-2}(p_i) \to \frac{n}{n-1}\left(\frac{|L\widetilde{W}_i|}{\tau}\right)^2(p_\infty),$$

This proves that

$$\widetilde{\omega}(p_\infty) + \frac{\epsilon}{2} \leqslant \left(\sqrt{\frac{n}{n-1}}\frac{|L\widetilde{W}_\infty|}{\tau}\right)^{\frac{1}{N}}(p_\infty),$$

which is a contradiction with (2.7).

For the remaining case; i.e., when there exists a sequence $(m_i) \in M$ s.t. $\widetilde{\omega}(m_i) - \frac{\epsilon}{2} > \widetilde{\theta}_i(m_i)$, $\widetilde{\omega} - \frac{\epsilon}{2}$ is not a subsolution to the rescaled Lichnerowicz equation on $B_\epsilon = \left\{m \in M : \ \widetilde{\omega}(m) - \frac{\epsilon}{2} > 0\right\}$ (here note that $\widetilde{\theta}_i > 0$, then $\widetilde{\omega}(m_i) - \frac{\epsilon}{2} > 0$ and $\widetilde{\omega} - \frac{\epsilon}{2} < \widetilde{\theta}_i$ on ∂B_ϵ if $B_\epsilon \subsetneq M$). By similar arguments to the first case, we also obtain a contradiction. □

The condition $\tau > 0$ plays an important role in the proof of the main theorem in [Dahl *et al.*, 2012] (or Theorem 3.1.1). Indeed, this condition implies that for any (φ, w) satisfying (1.2), we have

$$\varphi^N \leqslant C(g, \tau, \sigma)\max\{\|w\|_\infty, 1\}$$

(it is a consequence of the maximum principle), which plays a crucial role in the proof. When τ vanishes, this inequality does not remain true as shown by the following proposition:

Proposition 2.2.4. *Let $\tau : M \to \mathbb{R}$ be a C^0 function. For any $k > 1$, we denote by $\varphi_k > 0$ the unique solution to (1.2) associated to $w = k$. Assume that τ vanishes somewhere. Then*

$$\frac{\|\varphi_k\|_\infty^N}{k} \to \infty \quad \text{as } k \to \infty.$$

Proof. Set $\widetilde{\varphi}_k := \varphi_k / k^{\frac{1}{N}}$. Then $\widetilde{\varphi}_k$ is a solution to the following equation:

$$\frac{1}{k^{\frac{N-2}{N}}}\left(\frac{4(n-1)}{n-2}\Delta\widetilde{\varphi}_k + R\widetilde{\varphi}_k\right) + \frac{n-1}{n}\tau^2\widetilde{\varphi}_k^{N-1} = \frac{1}{\widetilde{\varphi}_k^{N+1}}. \tag{2.9}$$

Given $A > 0$, we set

$$\widetilde{\varphi}_A = \min\left\{\left(\frac{n}{(n-2)\tau^2}\right)^{\frac{1}{2N}}, A\right\}. \tag{2.10}$$

Fix $\epsilon > 0$ small enough. We first prove that

$$\widetilde{\varphi}_A \leqslant \widetilde{\varphi}_k + 2\epsilon, \quad \forall k \geqslant k_A, \tag{2.11}$$

for some k_A large enough depending on A. We proceed by contradiction. Assume that this is not true, so there exists a subsequence $\{m_k\} \in M$ s.t.

$$\widetilde{\varphi}_A(m_k) - 2\epsilon > \widetilde{\varphi}_k(m_k). \tag{2.12}$$

Next since $\widetilde{\varphi}_A \in C^0_+$, we can choose $\widetilde{\phi}_A \in C^2_+$ s.t.

$$|\widetilde{\phi}_A - \widetilde{\varphi}_A| \leqslant \epsilon/2. \tag{2.13}$$

Then it follows from (2.12) that

$$\widetilde{\phi}_A(m_k) - \epsilon > \widetilde{\varphi}_k(m_k). \tag{2.14}$$

Set $B_A = \left\{m \in M : \widetilde{\phi}_A - \epsilon > 0\right\}$. Since $\widetilde{\varphi}_k > 0$, we deduce from (2.14) that $\widetilde{\phi}_A - \epsilon$ is not a subsolution to (2.9) in B_A and hence there exists a sequence $\{p_k\} \in B_A$ s.t.

$$\frac{1}{k^{\frac{N-2}{N}}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\phi}_A - \epsilon\right)(p_k) + R(p_k)\left(\widetilde{\phi}_A - \epsilon\right)(p_k)\right] + \frac{n-1}{n}\tau^2(p_k)\left(\widetilde{\phi}_A - \epsilon\right)^{N-1}(p_k) > \frac{1}{\left(\widetilde{\phi}_A - \epsilon\right)^{N+1}(p_k)}$$

or equivalently,

$$\frac{\left(\widetilde{\phi}_A - \epsilon\right)^{N+1}(p_k)}{k^{\frac{N-2}{N}}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\phi}_A - \epsilon\right)(p_k) + R(p_k)\left(\widetilde{\phi}_A - \epsilon\right)(p_k)\right] + \frac{n-1}{n}\tau^2(p_k)\left(\widetilde{\phi}_A - \epsilon\right)^{2N}(p_k) > 1.$$

Taking $k \to \infty$ and assuming (after passing to a subsequence) $p_i \to p_\infty$, we obtain that

$$\frac{\left(\widetilde{\phi}_A - \epsilon\right)^{N+1}(p_k)}{k^{\frac{N-2}{N}}}\left[\frac{4(n-1)}{n-2}\Delta\left(\widetilde{\phi}_A - \epsilon\right)(p_k) + R(p_k)\left(\widetilde{\phi}_A - \epsilon\right)(p_k)\right] \to 0$$

and

$$\frac{n-1}{n}\tau^2\left(\widetilde{\phi}_A - \epsilon\right)^{2N}(p_k) \to \frac{n-1}{n}\tau^2(p_\infty)\left(\widetilde{\phi}_A - \epsilon\right)^{2N}(p_\infty),$$

This shows that
$$\frac{n-1}{n}\tau^2(p_\infty)\left(\widetilde{\phi}_A-\epsilon\right)^{2N}(p_\infty)\geqslant 1. \tag{2.15}$$
On the other hand, we have
$$\begin{aligned}\frac{n-1}{n}\tau^2(p_\infty)\left(\widetilde{\phi}_A-\epsilon\right)^{2N}(p_\infty) &\leqslant \frac{n-1}{n}\tau^2(p_\infty)\left(\widetilde{\varphi}_A-\frac{\epsilon}{2}\right)^{2N}(p_\infty) \qquad \text{(by (2.13))}\\ &\leqslant \frac{n-1}{n}\tau^2(p_\infty)\left(\widetilde{\varphi}_A^{2N}(p_\infty)-\left(\frac{\epsilon}{2}\right)^{2N}\right)\\ &\leqslant 1-\frac{n-1}{n}\tau^2(p_\infty)\left(\frac{\epsilon}{2}\right)^{2N}\\ &< 1,\end{aligned}$$
which is a contradiction with (2.15), and then (2.11) holds, as claimed. Now if $\widetilde{\varphi}_k \leqslant C$, we deduce from (2.11) that $\max \widetilde{\varphi}_A \leqslant C+2\epsilon$, which is false when $A\to+\infty$ since τ has some zeros. The proof is completed. □

We can be more precise. This is the content of the next proposition

Proposition 2.2.5. *Let $\tau: M\to\mathbb{R}$ be a C^0 function. We set*
$$L=\left\{(\varphi,w)\in W^{2,p}_+\times L^\infty : (\varphi,w) \textit{ satisfies } (1.2)\right\}.$$
Given $\alpha\geqslant\frac{1}{N}$, $\sup_{(\varphi,w)\in L}\frac{\|\varphi\|^N_{L^{N\alpha}}}{\max\{\|w\|_\infty,1\}}$ is bounded if and only if $|\tau|^{-\alpha}\in L^1$.

Proof. Applying Lemma 1.2.5 with $w_0=w$ and $w_1=\|w\|_\infty$, we have
$$\begin{aligned}\sup_{(\varphi,w)\in L}\frac{\|\varphi\|^N_{L^{N\alpha}}}{\max\{\|w\|_\infty,1\}} &= \sup_{\substack{(\varphi,w)\in L\\ w\text{ constant}}}\frac{\|\varphi\|^N_{L^{N\alpha}}}{\max\{|w|,1\}}\\ &=\sup_{k>1}\frac{\|\varphi_k\|^N_{L^{N\alpha}}}{k},\end{aligned}$$
where φ_k is the unique positive solution to (1.2) associated to $w=k$. Therefore, $\sup_{(\varphi,w)\in L}\frac{\|\varphi\|^N_{L^{N\alpha}}}{\max\{\|w\|_\infty,1\}}<\infty$ if and only if $\frac{\|\varphi_k\|^N_{L^{N\alpha}}}{k}$ is uniformly bounded for all $k>1$. Moreover note that with $C=C(g,\tau)$ large enough and not depending on k, $k^{\frac{1}{N}}/C$ is a subsolution to (1.2) associated to $w=k$, and hence for all $k>1$,
$$\varphi_k\geqslant\frac{k^{\frac{1}{N}}}{C}>\frac{1}{C}. \tag{2.16}$$
We first prove that $|\tau|^{-\alpha}\in L^1$ is a necessary condition. Set $\widetilde{\varphi}_k=\varphi_k/k^{\frac{1}{N}}$ and we let $\widetilde{\varphi}_A$ be given by (2.10). As in the proof of Proposition 2.2.4, we obtain that for all k large enough and depending on A,
$$\widetilde{\varphi}_A\leqslant\widetilde{\varphi}_k+\epsilon.$$
Assume that $\widetilde{\varphi}_k$ is uniformly bounded in $L^{N\alpha}$, so is $\widetilde{\varphi}_A$ by the previous inequality. On the other hand, it is clear that $\widetilde{\varphi}_A$ converges pointwise a.e to $\left(\frac{n}{n-1}\right)^{\frac{1}{2N}}|\tau|^{-\frac{1}{N}}$ as $A\to\infty$. Hence the monotone

convergence theorem implies that $|\tau|^{-\frac{1}{N}} \in L^{N\alpha}$, which is our claim.

We now prove that the condition is sufficient. Assume that $|\tau|^{-\alpha} \in L^1$. Multiplying (1.2) by $\varphi_k^{N\alpha+N+1}$ and integrating over M, we have

$$\frac{4(n-1)}{n-2}\int_M \varphi_k^{N\alpha+N+1}\Delta\varphi_k dv + \int_M R\varphi_k^{N\alpha+N+2}dv + \frac{n-1}{n}\int_M \tau^2\varphi_k^{N(\alpha+2)}dv = k^2\int_M \varphi_k^{N\alpha}dv. \quad (2.17)$$

As observed in Remark 1.2.4, $R\varphi_k^{N\alpha+N+2} + \frac{n-2}{n}\tau^2\varphi_k^{N(\alpha+2)}$ is uniformly bounded from below by a constant $\zeta = \zeta(g,\tau)$ which does not depend on k since we assume that $R \geqslant 0$ or $R = -\frac{n-1}{n}\tau^2$. Moreover, we have

$$\int_M \varphi_k^{N\alpha+N+1}\Delta\varphi_k dv = \frac{N\alpha+N+1}{(\frac{N\alpha+N}{2}+1)^2}\int_M |\nabla\varphi_k^{\frac{N\alpha+N}{2}+1}|^2 dv.$$

These facts combined with (2.16)-(2.17) lead to

$$\int_M \tau^2\varphi_k^{N(\alpha+2)dv} \leqslant C_1(C,\zeta)k^2\int_M \varphi_k^{N\alpha}dv. \quad (2.18)$$

On the other hand, we get that

$$\begin{aligned}\int_M \varphi_k^{N\alpha}dv &\leqslant \left(\int_M |\tau|^{-\alpha}dv\right)^{\frac{2}{\alpha+2}}\left(\int_M \tau^2\varphi_k^{N(\alpha+2)}dv\right)^{\frac{\alpha}{\alpha+2}} \quad \text{(by Hölder inequality)}\\ &\leqslant C_2(C_1,\tau,\alpha)\left(k^2\int_M \varphi_k^{N\alpha}dv\right)^{\frac{\alpha}{\alpha+2}} \quad \text{(by (2.18))}.\end{aligned}$$

It follows easily that for all $k > 1$

$$\frac{\|\varphi_k\|_{L^{N\alpha}}^N}{k} \leqslant C_2^{\frac{\alpha+2}{2\alpha}},$$

which completes our proof. □

The fixed point theorem above has some other consequences that we describe now. First, we have the following proposition.

Proposition 2.2.6. *Let data be given on M as specified in (4) and assume that (M,g) has no conformal Killing vector field and $\sigma \not\equiv 0$. If $\mathcal{Y}_g > 0$, then there exists a constant $\alpha = \alpha(g,\tau,\sigma) \in (0,1]$ such that the constraint equations w.r.t. the new data $(g,\alpha\tau,\sigma)$ admits a solution.*

Remark 2.2.7. *In the proof, we apply Leray-Schauder's Theorem 2.2.1 and not Corollary 2.2.2 as in the proof of Theorem 3.1.1.*

Proof. By Remark 1.2.4, we may assume $R > 0$. We construct a compact map $\widetilde{T} : L^\infty \times [0,1] \to L^\infty$ as follows. For each $(\varphi,t) \in L^\infty \times [0,1]$, there exists a unique $W_\varphi \in W^{2,p}$ s.t.

$$-\frac{1}{2}L^*LW_\varphi = \frac{n-1}{n}\varphi^N d\tau$$

and there exists a unique positive $\theta \in W^{2,p}$ satisfying

$$\frac{4(n-1)}{n-2}\Delta\theta + R\theta = -\frac{n-1}{n}t^{2N}\tau^2\theta^{N-1} + |\sigma + LW_\varphi|^2\theta^{-N-1}$$

(see [Dahl *et al.*, 2012, Lemma 2.2] and notice that $R > 0$). Then we define

$$\widetilde{T}(\varphi, t) = t\theta.$$

The continuity and compactness of $\widetilde{T}$ is clearly a direct consequence of the continuity and compactness of $T'(\varphi, t) := \frac{\widetilde{T}(\varphi,t)}{t} = \theta$.

Note that $T'(\varphi, t) = \widetilde{T}_1(G(\varphi), t)$. Here $G(\varphi) = |LW_\varphi + \sigma| \not\equiv 0$ and $\widetilde{T}_1 : L^\infty \times [0, 1] \to W^{2,p}_+$ is defined by $\widetilde{T}_1(w, t) = \theta$, where

$$\frac{4(n-1)}{n-2}\Delta\theta + R\theta = -\frac{n-1}{n}t^{2N}\tau^2\theta^{N-1} + w^2\theta^{-N-1}. \tag{2.19}$$

As proven in [Dahl *et al.*, 2012], G is continuous compact, so the continuity and compactness of T' and hence that of $\widetilde{T}$, will follow from the continuity of $\widetilde{T}_1$. Actually, we prove more: $\widetilde{T}_1$ is a C^1−map. Indeed, define $F : L^\infty \times [0, 1] \times W^{2,p}_+ \to L^{2p}$ by

$$F(w, t, \theta) = \frac{4(n-1)}{n-2}\Delta\theta + R\theta + \frac{n-1}{n}t^{2N}\tau^2\theta^{N-1} - w^2\theta^{-N-1}.$$

It is clear that F is continuous and $F(w, t, \widetilde{T}_1(w, t)) = 0$ for all $(w, t) \in L^\infty \times [0, 1]$. A standard computation shows that the Fréchet derivative of F w.r.t. θ is given by

$$F_\theta(w, t)(u) = \frac{4(n-1)}{n-2}\Delta u + Ru + \frac{(N-1)(n-1)}{n}t^{2N}\tau^2\theta^{N-2}u + (N+1)w^2\theta^{-N-2}u.$$

We first note that $F_\theta \in C\left(L^\infty \times [0, 1], L(W^{2,p}, L^p)\right)$, where $L(W^{2,p}, L^p)$ denotes the Banach space of all linear continuous maps from $W^{2,p}$ into L^p. Now, given $(w_0, t_0) \in L^\infty \times [0, 1]$, setting $\theta_0 = \widetilde{T}_1(w_0, t_0)$, we have

$$F_{\theta_0}(w_0, t_0)(u) = \frac{4(n-1)}{n-2}\Delta u + \left(R + \frac{(N-1)(n-1)}{n}t_0^{2N}\tau_0^2\theta_0^{N-2} + (N+1)w_0^2\theta_0^{-N-2}\right)u.$$

Since

$$R + \frac{(N-1)(n-1)}{n}t_0^{2N}\tau_0^2\theta_0^{N-2} + (N+1)w_0^2\theta_0^{-N-2} \geqslant \min R > 0,$$

we conclude that $F_{\theta_0}(w_0, t_0) : W^{2,p} \to L^p$ is an isomorphism. The implicit function theorem then implies that $\widetilde{T}_1$ is a C^1 function in a neighborhood of (w_0, t_0), which proves our claim.

Next applying Leray-Schauder's Theorem 2.2.1 to $\widetilde{T}$, we obtain as a direct consequence that there exist $\varphi_0 \in L^\infty$ and $t_0 \in (0, 1]$ s.t.

$$\begin{aligned}
\frac{4(n-1)}{n-2}\Delta\theta_0 + R\theta_0 &= -\frac{n-1}{n}t_0^{2N}\tau^2\theta_0^{N-1} + |\sigma + LW_0|^2\theta_0^{-N-1} \\
-\frac{1}{2}L^*LW_0 &= \frac{n-1}{n}\varphi_0^N d\tau,
\end{aligned}$$

with $\varphi_0 = t_0\theta_0 \in W^{2,p}$. Indeed, set

$$K = \left\{\varphi \in L^\infty | \ \exists t \in [0,1] \text{ such that } \varphi = \widetilde{T}(\varphi, t)\right\}.$$

It is clear that $\widetilde{T}(\varphi, 0) = 0$ for all $\varphi \in L^\infty$. Assume that such (φ_0, t_0) does not exist. Then $K = \{0\}$. By Leray-Schauder's Theorem 2.2.1, there exists φ s.t. $\varphi = \widetilde{T}(\varphi, 1) = T(\varphi)$ which belongs to K. So $\varphi = 0$ which is impossible since $T(\varphi) \not\equiv 0$.

Now replacing φ_0 by $t_0\theta_0$ in the vector equation, we get that (θ_0, W_0) is a solution to (3) w.r.t. the new data $(g, \alpha\tau, \sigma)$, with $\alpha = t_0^N$. □

Proposition 2.2.6 is a direct consequence of the small-TT case (i.e. a smallness assumption on the transverse-traceless tensor) in [Holst *et al.*, 2009] and [Maxwell, 2009]. More precisely, we can easily check the following, which is developed further in [Gicquaud et Ngô, 2014].

Remark 2.2.8. (φ, W) *is a solution to the conformal equations w.r.t. the seed data* (g, τ, σ) *if and only if* $(C^{-1}\varphi, C^{-\frac{N+2}{2}}W)$ *is a solution to the conformal equations w.r.t. the data* $(g, C^{\frac{N-2}{2}}\tau, C^{-\frac{N+2}{2}}\sigma)$ *for all constant* $C > 0$.

Proposition 2.2.9 (see [Holst *et al.*, 2009] or [Maxwell, 2009])**.** *Let data be given on* M *as specified in* (4) *associated to the vacuum case. Assume that* $\mathcal{Y}_g > 0$, (M, g) *has no conformal Killing vector field and* $\sigma \not\equiv 0$. *If* $\|\sigma\|_{L^\infty}$ *is small enough (depending only on g and* τ*), then the system* (3) *has a solution* (φ, W).

From Remark 2.2.8, with $C = \alpha^{-\frac{2}{N-2}}$, Proposition 2.2.6 is equivalent to the fact that (3) w.r.t. the new data $(g, \tau, \alpha^{\frac{N+2}{N-2}}\sigma)$ admits a solution, and this holds for α small enough by Proposition 2.2.9.

In particular, this approach has the advantage to give an unifying point of view of the limit equation method in [Dahl *et al.*, 2012] and the far-from CMC results in [Gicquaud et Ngô, 2014], [Holst *et al.*, 2009] and [Maxwell, 2009].

The main theorem in [Dahl *et al.*, 2012] (or Theorem 3.1.1) says that the nonexistence of non-trivial solutions to the limit equation (2.2) implies the existence of a solution to (3). The opposite question naturally arises whether the existence of a solution to (3) implies the nonexistence of (non-trivial) solution to the limit equation. The following proposition shows that this is false.

Proposition 2.2.10. *There exists seed data* (M, g, τ, σ) *such that both the corresponding* (3) *and* (2.2) *admit (non-trivial) solutions.*

Proof. In [Dahl *et al.*, 2012], Dahl–Gicquaud–Humbert prove that there exist (M, g, τ, σ) and $\alpha_0 \in (0, 1]$ s.t. $\mathcal{Y}_g > 0$ and the corresponding limit equation

$$-\frac{1}{2}L^*LW = \alpha_0\sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{\tau}$$

admits a nontrivial solution $W \in W^{2,p}$ (see [Dahl *et al.*, 2012, Proposition 1.6]). Now note that for all $\alpha > 0$,

$$\frac{d\alpha\tau}{\alpha\tau} = \frac{d\tau}{\tau}.$$

so the limit equation for the 4-tuple $(M, g, \alpha\tau, \sigma)$ also admits a non-trivial solution. Taking α given by Proposition 2.2.6 provides $(M, g, \alpha\tau, \sigma)$ as desired. □

2.3 Half-Continuous Maps and Applications

In this section we introduce the theory of half-continuous maps and its applications to solving the constraint equations. We summarize results on half-continuous maps in the next subsection. For the proofs we refer the reader to [Bich, 2006] or [Termwuttipong et Kaewtem, 2010].

2.3.1 Half-Continuous Maps

Definition 2.3.1. *Let C be a subset of a Banach space X. A map $f : C \to X$ is said to be half-continuous if for each $x \in C$ with $x \neq f(x)$ there exists $p \in X^*$ and a neighborhood W of x in C such that*

$$\langle p, f(y) - y \rangle > 0$$

for all $y \in W$ with $y \neq f(y)$.

The following proposition gives a relation between half-continuity and continuity.

Proposition 2.3.2 (see [Termwuttipong et Kaewtem, 2010], Proposition 3.2)**.** *Let X be a Banach space and C be a subset of X. Then every continuous map $f : C \to X$ is half-continuous.*

Remark 2.3.3 (see [Termwuttipong et Kaewtem, 2010])**.** *There are some half-continuous maps which are not continuous. For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 3 & \text{if } x \in [0, 1), \\ 2 & \text{otherwise.} \end{cases}$$

Then f is half-continuous but not continuous.

Theorem 2.3.4 (see [Termwuttipong et Kaewtem, 2010], Theorem 3.9 or [Bich, 2006], Theorem 3.1)**.** *Let C be a nonempty compact convex subset of a Banach space X. If $f : C \to C$ is half-continuous, then f has a fixed point.*

A direct consequence of Theorem 2.3.4 is the following corollary, which is our main tool in the next subsection.

Corollary 2.3.5. *Let C be a nonempty closed convex subset of a Banach space X. If $f : C \to C$ is half-continuous and $f(C)$ is precompact, then f has a fixed point.*

Proof. Since $\overline{f(C)}$ is nonempty compact and X is a Banach space, $\overline{\operatorname{conv}}(f(C))$ is a nonempty compact convex subset of X (see [Rudin, 1991], Theorem 3.20). Moreover, since C is a closed convex subset of X and $f(C) \subset C$, we have $\overline{\operatorname{conv}}(f(C)) \subset C$, and hence $f\left(\overline{\operatorname{conv}}(f(C))\right) \subset f(C) \subset \overline{\operatorname{conv}}(f(C))$. Now restricting f to $\overline{\operatorname{conv}}(f(C))$ and applying the previous theorem, we obtain the desired conclusion. □

2.3.2 Existence Results for Modified Constraint Equations

Here we apply the concept of half-continuity to improve recent existence results for (3) (see [Holst *et al.*, 2009] or [Maxwell, 2009]).

The first non-CMC result for (3) is the near-CMC case, which is presented by many authors: if $\frac{\max|d\tau|}{\min|\tau|}$ is small enough, then (3) admits a solution (see [Bartnik et Isenberg, 2004]). Recently, Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012] improved this result. They show that (3) has a solution, provided $\|\frac{d\tau}{\tau}\|_{L^n}$ is small enough (see [Dahl *et al.*, 2012, Corollary 1.3 and 14]). However, for a smooth vanishing τ, these assumptions never hold. Therefore, we treat a generalization of (3), with $d\tau$ replaced by a 1-form $\xi \in L^\infty$ in the vector equation. Namely, let data be given on M as specified in (4) and choose also a $1-$form $\xi \in L^\infty$. We are interested in the following system.

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |\sigma + LW|^2\varphi^{-N-1} \tag{2.21a}$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N\xi. \tag{2.21b}$$

Note that all the methods described above can be applied in this context when $\tau > 0$. A natural question is then whether this coupled nonlinear elliptic system has a solution under a similar condition; i.e., $\|\frac{\xi}{\tau}\|_{L^n}$ is small enough. As τ vanishes, it becomes more complicated to apply the method of global supersolution introduced by Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009] because the construction of a supersolution to the Lichnerowicz equation seems to fail with their method near the zero set of τ, which from now on is denoted by $Z(\tau)$. Before going further, we establish a useful estimate for (3).

Let $\mathcal{I}$ be the family of all solutions of (3) associated to the vacuum case for fixed given data (g, τ, σ). Provided $\tau > 0$, it was obtained in [Dahl *et al.*, 2012] by induction that there exists a positive constant $C = C(M, g, \tau, \sigma)$ s.t.

$$\|\varphi\|_\infty \leqslant C\max\{\|LW\|_{L^2}^{\frac{1}{N}}, 1\}, \quad \forall(\varphi, W) \in \mathcal{I}.$$

For a vanishing τ, there is no reason to get the estimate above. However, by a slight change in the proof, we have the following proposition.

Proposition 2.3.6. *Let data be given on M as specified in (4) associated to the vacuum case and assume that (5) holds. Assume further that $Z(\tau)$ has zero Lebesgue measure if $\mathcal{Y}_g \leqslant 0$. Given $l > 0$, there exists a positive constant $C = C(M, g, \sigma, \tau, l)$ s.t. for any $(\varphi, W) \in \mathcal{I}$ satisfying $\|LW\|_{L^2} \leqslant l$ we have*

$$\|\varphi\|_\infty \leqslant C.$$

Proof. For simplicity, we assume that $\tau \in C^1(M)$. We begin with the observation that, to prove the proposition, it suffices to show that there exists a constant $c = c(n, g, \tau, \sigma, l) > 0$ s.t. for any $(\varphi, W) \in \mathcal{I}$ satisfying $\|LW\|_{L^2} \leqslant l$ we have $\|LW\|_\infty < c$. In fact, assume that this is true. Then, from Lemma 1.2.5, we have that $\varphi \leqslant \varphi_c$, where φ_c is a unique positive solution to the Lichnerowicz equation (1.2) associated to $w = c + \|\sigma\|_\infty$, and hence taking $C = \max \varphi_c$, the proposition follows.

Now we will prove the boundedness of $\|LW\|_\infty$ as mentioned above. Set $q_i = 2\left(\frac{N+2}{4}\right)^i$ for all $i \in \mathbb{N}$. We first show inductively that if $|LW|$ is uniformly bounded in L^{q_i}−norm by $r_i > 0$, then $|LW|$ is bounded in $L^{q_{i+1}}$ by $r_{i+1} = r_{i+1}(n, g, \tau, \sigma, q_i, r_i) > 0$. In fact, multiplying the Lichnerowicz equation by $\varphi^{\frac{(N+2)q_i}{2}-1}$ and integrating over M, we have

$$\begin{aligned}
\frac{4(n-1)}{n-2}\int_M \varphi^{\frac{(N+2)q_i}{2}-1}\Delta\varphi dv + \int_M R\varphi^{\frac{(N+2)q_i}{2}} dv+&\frac{n-1}{n}\int_M \tau^2\varphi^{N+\frac{(N+2)q_i}{2}-2}dv\\
&= \int_M |\sigma + LW|^2\varphi^{\frac{(N+2)(q_i-2)}{2}} dv\\
&\leqslant \|\sigma + LW\|^2_{L^{q_i}}\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}}\\
&\qquad \text{(by } q_i \geqslant 2 \text{ and Hölder inequality)}\\
&\leqslant 2\left(\|\sigma\|^2_{L^{q_i}} + \|LW\|^2_{L^{q_i}}\right)\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}}.
\end{aligned} \tag{2.22}$$

Since

$$\int_M \varphi^{\frac{(N+2)q_i}{2}-1}\Delta\varphi dv = \frac{8\left((N+2)q_i-2\right)}{(N+2)^2q_i^2}\int_M |\nabla\varphi^{\frac{(N+2)q_i}{4}}|^2 dv \geqslant 0, \tag{2.23}$$

and since the term $\int_M R\varphi^{\frac{(N+2)q_i}{2}} dv + \frac{n-1}{n}\int_M \tau^2\varphi^{N+\frac{(N+2)q_i}{2}-2}dv$ is uniformly bounded from below as observed in Remark 1.2.4. we obtain from (2.22) that

$$\int_M |\nabla\varphi^{\frac{(N+2)q_i}{4}}|^2 dv \leqslant c_1(g, \tau, q_i) + c_2(g, \tau, \sigma, q_i, r_i)\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}},$$

and then

$$\begin{aligned}
\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^N} &\leqslant c_3(M, g)\left(\|\nabla\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2} + \|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}\right) \quad \text{(by the Sobolev inequality)}\\
&\leqslant c_3\left(c_1 + c_2\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}^{\frac{2(q_i-2)}{q_i}} + \|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}\right).
\end{aligned} \tag{2.24}$$

To show that $\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^N}$ is bounded, by (2.24) it suffices to assume that

$$\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^N} \leqslant 3c_3\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2} \tag{2.25}$$

and to prove that $\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}$ is bounded. We study two cases.

- Case 1. $\mathcal{Y}_g > 0$*:* By Remark 1.2.4, we can assume that $R > 0$ and then it is clear from (2.22)-(2.23) that

$$\int_M \varphi^{\frac{(N+2)q_i}{2}} dv \leqslant \frac{2}{\min R}\left(\|\sigma\|^2_{L^{q_i}} + r_i^2\right)\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}},$$

which implies the boundedness of $\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}$.

- Case 2. $\mathcal{Y}_g \leqslant 0$: Given $k > 0$, we define

$$B_k = \left\{ m \in M : \quad \varphi^{\frac{(N+2)q_i}{4}}(m) \geqslant \frac{1}{k}\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2} \right\}.$$

Let χ_{B_k} denote the characteristic function of B_k. We have

$$\begin{aligned}
1 = \int_M \frac{\varphi^{\frac{(N+2)q_i}{2}}}{\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}} dv &\leqslant \int_M \frac{\chi_{B_k}\varphi^{\frac{(N+2)q_i}{2}}}{\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}} dv + \int_{M\setminus B_k} \frac{\varphi^{\frac{(N+2)q_i}{2}}}{\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}} dv \\
&\leqslant \frac{\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^N}}{\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2}} Vol(B_k)^{\frac{N-2}{N}} + \frac{1}{k^2} Vol(M \setminus B_k) \\
&\qquad \text{(by Hölder inequality and the definition of } B_k\text{)} \\
&\leqslant 9c_3^2 Vol(B_k)^{\frac{N-2}{N}} + \frac{1}{k^2} Vol(M) \quad \text{(by (2.25))}.
\end{aligned}$$

Taking $k_0 \geqslant 2Vol(M)+1$, it follows that $Vol(B_{k_0}) \geqslant 2c_4(n, c_3) > 0$. On the other hand, since $Z(\tau)$ is a closed, subset of M with zero measure, there exists a neighborhood B_i of $Z(\tau)$, depending on c_4 s.t. $Vol(B_i) \leqslant c_4$. Next we get by (2.22)-(2.23) that

$$\int_M R\varphi^{\frac{(N+2)q_i}{2}} dv + \frac{n-1}{n}\int_{B_{k_0}\setminus B_i} \tau^2 \varphi^{N+\frac{(N+2)q_i}{2}-2} dv \leqslant 2\left(\|\sigma\|^2_{L^{q_i}} + r_i^2\right)\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}}. \tag{2.26}$$

Set $\tau_i = \inf_{M\setminus B_i}|\tau| > 0$. Since $\varphi^{\frac{(N+2)q_i}{4}} \geqslant \frac{1}{k_0}\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}$ on B_{k_0} and since $Vol\,(B_{k_0} \setminus B_i) \geqslant c_4$, it follows from (2.26) that

$$-\|R\|_{L^\infty}\|\varphi^{\frac{(N+2)q_i}{4}}\|^2_{L^2} + \frac{n-1}{n} c_4 \tau_i^2 \left(\frac{\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}}{k_0}\right)^{2\left(\frac{(q_i+2)(N+2)-8}{q_i(N+2)}\right)} \leqslant 2\left(\|\sigma\|^2_{L^{q_i}} + r_i^2\right)\left(\int_M \varphi^{\frac{(N+2)q_i}{2}} dv\right)^{\frac{q_i-2}{q_i}}.$$

Since

$$\frac{q_i-2}{q_i} < 1 < \frac{(q_i+2)(N+2)-8}{q_i(N+2)}$$

for all $i \in \mathbb{N}$, we get from the previous inequality that $\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2}$ is bounded by $c_5 = c_5(n, g, \tau, \sigma, r_i, c_4, k_0, q_i)$.

In both cases, we have showed that $\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^2} \leqslant c_5$ and hence by (2.25) that

$$\|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^N} \leqslant c_6(c_5, c_3). \tag{2.27}$$

Now by the Sobolev embedding theorem, from the vector equation, there exists $c_7 = c_7(M, g)$ s.t.

$$\begin{aligned} \|LW\|_{L^{\frac{nq_i(N+2)}{(4n-(N+2)q_i)^+}}} &\leqslant c_7 \|\varphi^N d\tau\|_{L^{\frac{(N+2)q_i}{4}}} \\ &\leqslant c_7 \|d\tau\|_\infty \|\varphi^N\|_{L^{\frac{(N+2)q_i}{4}}} \quad \left(\text{since } \tau \in C^1\right) \\ &\leqslant c_8(c_7, \tau) \|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^N}^{\frac{4N}{(N+2)q_i}} \\ &\leqslant c_9(c_8, c_6) \qquad\qquad (\text{by } (2.27)). \end{aligned} \tag{2.28}$$

Here $(4n - (N+2)q_i)^+ = \max\{4n - (N+2)q_i, 0\}$ and $L^{\frac{nq_i(N+2)}{(4n-(N+2)q_i)^+}}$ is understood to be L^∞ if $4n \leqslant (N+2)q_i$. Since

$$q_{i+1} < \frac{nq_i(N+2)}{(4n - (N+2)q_i)^+},$$

it follows from (2.28) that $\|LW\|_{L^{q_{i+1}}} \leqslant r_{i+1}(n, g, \tau, \sigma, q_i, r_i)$ as claimed.

Finally, note that $\frac{N+2}{4} > 1$. We can then take i_0 large enough depending only on n s.t. $q_{i_0} \geqslant \left[\frac{4n}{N+2}\right] + 1$. Thus, applying inductively (2.28) for $i \leqslant i_0$, provided $\|LW\|_{L^{q_0}} = \|LW\|_{L^2} \leqslant l$, we obtain that $|LW|$ is uniformly bounded in L^∞ by $c = c(n, g, \tau, \sigma, l) > 0$, which completes our proof. □

We are now ready to prove the second main result of this chapter.

Theorem 2.3.7 (Near-CMC)**.** *Assume that $\tau \in L^\infty$, $\xi \in L^\infty$, $g \in W^{2,p}$ $(p > n)$, (M, g) has no conformal Killing vector field, and $\sigma \not\equiv 0$ if $\mathcal{Y}_g \geqslant 0$. Assume further that $Z(\tau)$ has zero Lebesgue measure if $\mathcal{Y}_g \leqslant 0$. If $\|\frac{\xi}{\tau}\|_{L^n}$ is small enough, then the system of equations (2.21) admits a solution (φ, W).*

Proof. Recall that T, defined in Section 2.2 (where $d\tau$ is replaced by ξ in the vector equation), is a continuous compact map and $T(\varphi) > 0$ for all $\varphi \in C^0$. As explained in Remark 1.2.4, there exists a constant $\kappa_1 = \kappa_1(g, \tau)$ s.t.

$$RT(\varphi)^{N+2} + \frac{n-2}{n}\tau^2 T(\varphi)^{2N} \geqslant \kappa_1, \quad \forall \varphi \in C^0. \tag{2.29}$$

Set $\kappa = \max\left\{|\kappa_1|, \int_M |\sigma|^2 dv\right\}$. Let S be given by

$$S(\varphi) = \begin{cases} \min\{T(\varphi), a\} & \text{if } \|LW_\varphi\|_{L^2} \leqslant \sqrt{\kappa}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.30}$$

and set $\mathscr{C} = \left\{\varphi \in C^0 : 0 \leqslant \varphi \leqslant a\right\}$, where a will be determined later.

Since T is a continuous compact map from C^0 to C^0_+ and since by definition $0 \leqslant S(\varphi) \leqslant a$ for all $\varphi \in \mathscr{C}$, S maps $\mathscr{C}$ into itself and $S(\mathscr{C})$ is precompact. Assume for the moment that the half-continuity of S is proven. By Corollary 2.3.5, S has a fixed point φ_0. Note that φ_0 is not zero; otherwise $0 = \varphi_0 = S(\varphi_0)$, hence $\|LW_{\varphi_0}\|_{L^2} = 0 \leqslant \sqrt{\kappa}$. We get from the definition of S that $S(\varphi_0) = \min\{T(\varphi_0), a\} > 0$ which is a contradiction with $S(\varphi_0) = 0$. Since $\varphi_0 \not\equiv 0$, so is $S(\varphi_0)$, the definition of S implies that $\|LW_{\varphi_0}\|_{L^2} \leqslant \sqrt{\kappa}$ and

$$\varphi_0 = \min\{T(\varphi_0), a\} \leqslant T(\varphi_0). \tag{2.31}$$

Set

$$K = \left\{\varphi :\ \|LW_\varphi\|_{L^2} \leqslant \sqrt{\kappa} \ \text{ and } \ \varphi \leqslant T(\varphi)\right\}.$$

Arguing as in the proof of Theorem 2.3.6, we obtain that if any $\varphi \in K$ satisfies $\|LW_\varphi\|_{L^{q_i}} \leqslant r_i$ for some constant $r_i > 0$, then

$$\left\| T(\varphi)^{\frac{(N+2)q_i}{4}} \right\|_{L^N} \leqslant \tilde{r}_i(n, g, \tau, \sigma, r_i, q_i), \tag{2.32}$$

where $q_i = 2\left(\frac{N+2}{4}\right)^i$ for all $i \in \mathbb{N}$. Therefore, by the Sobolev embedding theorem, we have from the vector equation that

$$\begin{aligned}
\|LW_\varphi\|_{L^{\frac{nq_i(N+2)}{(4n-(N+2)q_i)^+}}} &\leqslant r(M,g)\|\varphi^N\xi\|_{L^{\frac{(N+2)q_i}{4}}} \\
&\leqslant r\|\xi\|_\infty \|\varphi^N\|_{L^{\frac{(N+2)q_i}{4}}} && \text{(since } \xi \in L^\infty\text{)} \\
&\leqslant r\|\xi\|_\infty \|\varphi^{\frac{(N+2)q_i}{4}}\|_{L^N}^{\frac{4N}{(N+2)q_i}} \\
&\leqslant r\|\xi\|_\infty \left\| T(\varphi)^{\frac{(N+2)q_i}{4}} \right\|_{L^N}^{\frac{4N}{(N+2)q_i}} && \text{(by } \varphi \leqslant T(\varphi)\text{)} \\
&\leqslant r_{i+1}(\xi, r, \tilde{r}_i) && \text{(by (2.32))},
\end{aligned} \tag{2.33}$$

where $(4n-(N+2)q_i)^+ = \max\{4n-(N+2)q_i, 0\}$ and $L^{\frac{nq_i(N+2)}{(4n-(N+2)q_i)^+}}$ is understood to be L^∞ if $4n \leqslant (N+2)q_i$. As in the proof of Theorem 2.3.6, we obtain inductively from (2.33) that for all $\varphi \in K$, there exists a constant $C = C(n, g, \tau, \xi, \kappa) > 0$ s.t.

$$\|LW_\varphi\|_{L^\infty} \leqslant C,$$

and hence by Lemma 1.2.5 the set $T(K)$ is bounded by $\max \theta_C$, where θ_C is the unique positive solution to the Lichnerowicz equation (1.2) associated to $w = \|\sigma\|_{L^\infty} + C$. Thus, taking $a = \max \theta_C + 1$, since $\varphi_0 \in K$, we also obtain from (2.31) that $\varphi_0 = T(\varphi_0)$, which proves the theorem.

We now prove the half-continuity of S. Since T is continuous, so is S at φ satisfying $\|LW_\varphi\|_{L^2} \neq \sqrt{\kappa}$. For φ s.t. $\|LW_\varphi\|_{L^2} = \sqrt{\kappa}$, multiplying the Lichnerowicz equation by $T(\varphi)^{N+1}$ and integrating over M, we have

$$\begin{aligned}
\frac{4(n-1)(N+1)}{(n-2)\left(\frac{N}{2}+1\right)^2}\int_M |\nabla T(\varphi)^{\frac{N+2}{2}}|^2 dv + \int_M RT(\varphi)^{N+2} dv + \frac{n-1}{n}\int_M \tau^2 T(\varphi)^{2N} dv &= \int_M |\sigma + LW_\varphi|^2 dv \\
&= \int_M |\sigma|^2 dv + \int_M |LW_\varphi|^2 dv \\
&= \int_M |\sigma|^2 dv + \kappa.
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_M \tau^2 T(\varphi)^{2N} dv &\leqslant n\left(\int_M |\sigma|^2 dv + \kappa - \left(\int_M RT(\varphi)^{N+2} dv + \frac{n-2}{n}\int_M \tau^2 T(\varphi)^{2N} dv\right)\right)\\ &\leqslant n\left(\int_M |\sigma|^2 dv + \kappa + |\kappa_1|\right) \qquad \text{(by (2.29))}\\ &\leqslant 3n\kappa.\end{aligned} \tag{2.34}$$

On the other hand, we get from the vector equation that

$$\begin{aligned}\kappa = \int_M |LW_\varphi|^2 dv &\leqslant C_5(g)\left\|W_\varphi\right\|^2_{W^{2,\frac{2n}{n+2}}} \qquad \text{(by Sobolev imbedding)}\\ &\leqslant C_6(g, C_5)\|L^* LW_\varphi\|^2_{L^{\frac{2n}{n+2}}}\\ &\leqslant C_7(C_6)\left(\int_M |\xi|^{\frac{2n}{n+2}} \varphi^{\frac{2nN}{n+2}} dv\right)^{\frac{n+2}{n}}\\ &\leqslant C_7\left\|\frac{\xi}{\tau}\right\|^2_{L^n}\int_M \tau^2\varphi^{2N} dv \quad \text{(by Hölder inequality).}\end{aligned} \tag{2.35}$$

By (2.34) and (2.35), we obtain that

$$\int_M \tau^2 T(\varphi)^{2N} dv \leqslant 3nC_7\left\|\frac{\xi}{\tau}\right\|^2_{L^n}\int_M \tau^2\varphi^{2N} dv.$$

If $\left\|\frac{\xi}{\tau}\right\|_{L^n}$ is small enough s.t. $3nC_7\left\|\frac{\xi}{\tau}\right\|^2_{L^n} < 1$, it follows from the previous inequality that there exists $m \in M$ s.t. $0 < T(\varphi)(m) < \varphi(m)$ (note that $T(\varphi) \in C^0_+$). Therefore, since T is continuous, there exists $\delta = \delta(\varphi) > 0$ small enough s.t.

$$0 < T(\theta)(m) < \theta(m), \quad \forall\theta \in B(\varphi, \delta) \cap \mathscr{C},$$

and hence from the fact that

$$-(S(\theta)(m) - \theta(m)) = \begin{cases} -(\min\{T(\theta)(m), a\} - \theta(m)) & \text{if } \|LW_\theta\|_{L^2} \leqslant \sqrt{\kappa},\\ \theta(m) & \text{otherwise,}\end{cases}$$

we conclude that

$$-(S(\theta)(m) - \theta(m)) > 0 \tag{2.36}$$

for all $\theta \in B(\varphi, \delta) \cap \mathscr{C}$.

Now let $p : C^0 \longrightarrow \mathbb{R}$ be defined by $p(f) = -f(m)$ for all $f \in C^0$. It is obvious that $p \in \left(C^0\right)^*$. Moreover, Inequality (2.36) tells us that $p(S(\theta) - \theta) > 0$ for all $\theta \in B(\varphi, \delta) \cap \mathscr{C}$, and then by definition S is half-continuous at φ as claimed. The proof is completed. □

Our next existence result deals with the far-from-CMC case. It makes progress compared with the statements of Holst–Nagy–Tsogtgerel [Holst *et al.*, 2009] and Maxwell [Maxwell, 2009] (see Proposition 2.2.9), where the smallness assumption on σ is in L^∞. Here our assumption is on the L^2-norm of σ.

Theorem 2.3.8 (**Far-from-CMC**). *Let data be given on M as specified in (4) associated to the vacuum case. Assume that $\mathcal{Y}(g) > 0$, (M, g) has no conformal Killing vector field and $\sigma \not\equiv 0$. If $\|\sigma\|_{L^2}$ is small enough (depending only on g and τ), then the system (3) has a solution (φ, W).*

Proof. Regarding Remark 1.2.4, we may assume that $R > 0$. We define

$$S(\varphi) = \begin{cases} \min\{T(\varphi), a\} & \text{if } \frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi^{\frac{N(N+2)}{2}} dv\right)^{\frac{2}{N}} \leqslant 2\int_M |\sigma|^2 dv \\ 0 & \text{otherwise,} \end{cases} \tag{2.37}$$

where a is to be determined later. Let

$$\mathscr{C} = \left\{\varphi \in C^0(M) : \ \|\varphi\|_\infty \leqslant a\right\}.$$

As in the previous proof, S maps $\mathscr{C}$ into itself and $S(\mathscr{C})$ is precompact since T is a compact map from C^0 into C^0_+. Assume that the half-continuity of S is proven. Then Corollary 2.3.5 implies that S admits a fixed point φ_0. Note that φ_0 is not zero. Indeed, if $0 = \varphi_0 = S(\varphi_0)$, it follows that $\frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi_0^{\frac{N(N+2)}{2}} dv\right)^{\frac{2}{N}} = 0 \leqslant 2\int_M |\sigma|^2 dv$, and hence from the definition of S we get that $S(\varphi_0) = \min\{T(\varphi_0), a\} > 0$ which is a contradiction with $S(\varphi_0) = 0$. Since $\varphi_0 \not\equiv 0$, so is $S(\varphi_0)$, and the definition of S implies that

$$\frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi_0^{\frac{N(N+2)}{2}} dv\right)^{\frac{2}{N}} \leqslant 2\int_M |\sigma|^2 dv \quad \text{and} \quad \varphi_0 = S(\varphi_0) = \min\{T(\varphi_0), a\} \leqslant T(\varphi_0).$$

On the other hand, the first condition on φ_0 and the smallness assumption on $\|\sigma\|_{L^2}$ implies that

$$\int_M |LW_{\varphi_0}|^2 dv \leqslant \int_M |\sigma|^2 dv.$$

Indeed,

$$\begin{aligned} \int_M |LW_{\varphi_0}|^2 dv &\leqslant C(g)\|\varphi_0^N d\tau\|^2_{L^{\frac{2n}{n+2}}} \quad \text{(by Sobolev imbedding theorem)} \\ &\leqslant C\|d\tau\|^2_{L^p}\left(\int_M \varphi_0^{\frac{2nNp}{(n+2)p-2n}} dv\right)^{\frac{(n+2)p-2n}{np}} \quad \text{(by Hölder inequality)} \\ &\leqslant C\|d\tau\|^2_{L^p}\left(\int_M \varphi_0^{\frac{N(N+2)}{2}} dv\right)^{\frac{4}{N+2}} \quad \text{(by Hölder inequality and } p > n) \\ &\leqslant C\|d\tau\|^2_{L^p}\left(\frac{(N+2)^2}{2(N+1)\mathcal{Y}_g}\right)^{\frac{2N}{N+2}} \|\sigma\|_{L^2}^{\frac{2(N-2)}{N+2}} \int_M |\sigma|^2 dv \quad \text{(by the first condition on } \varphi_0) \\ &\leqslant \int_M |\sigma|^2 dv, \end{aligned} \tag{2.38}$$

where the last inequality holds provided $\|\sigma\|_{L^2}$ is small enough so that $C\|d\tau\|^2_{L^p}\left(\frac{(N+2)^2}{2(N+1)\mathcal{Y}_g}\right)^{\frac{2N}{N+2}} \|\sigma\|_{L^2}^{\frac{2(N-2)}{N+2}} \leqslant 1$. Setting

$$K = \left\{\varphi : \ \|LW_\varphi\|_{L^2} \leqslant \|\sigma\|_{L^2} \ \text{ and } \ \varphi \leqslant T(\varphi)\right\},$$

similarly to the proof of Theorem 2.3.7, we then obtain that $T(K)$ is uniformly bounded in L^∞ by $C = C(g,\tau,\sigma)$. Thus, taking $a \geqslant C$, since $\varphi_0 \in K$, we obtain from the second condition on φ_0 that $\varphi_0 = T(\varphi_0)$, which completes our proof.

Now we prove the half-continuity of S on $\mathscr{C}$. Since T is continuous, so is S at φ satisfying

$$\frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} \neq 2\int_M |\sigma|^2 dv.$$

For the remaining φ; i.e., when $\frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} = 2\int_M |\sigma|^2 dv$, first note that, arguing as done to get (2.38), we have

$$\int_M |LW_\varphi|^2 dv \leqslant \int_M |\sigma|^2 dv. \tag{2.39}$$

Next we prove that there exists $m \in M$ s.t. $\varphi(m) > T(\varphi)(m)$. We argue by contradiction. Assume that it is not true; then

$$\begin{aligned}\frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} &\geqslant \frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \varphi^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} = 2\int_M |\sigma|^2 dv \\ &\geqslant \int_M |\sigma|^2 dv + \int_M |LW|^2 dv \quad (\text{by } (2.39)).\end{aligned} \tag{2.40}$$

On the other hand, multiplying the Lichnerowicz equation by $T(\varphi)^{N+1}$ and integrating over M, we obtain

$$\frac{16(n-1)(N+1)}{(n-2)(N+2)^2}\int_M |\nabla T(\varphi)^{\frac{N+2}{2}}|^2 dv + \int_M RT(\varphi)^{N+2}dv + \frac{n-1}{n}\int_M \tau^2 T(\varphi)^{2N}dv = \int_M |\sigma|^2 dv + \int_M |LW_\varphi|^2 dv. \tag{2.41}$$

Since

$$\begin{aligned}&\frac{16(n-1)(N+1)}{(n-2)(N+2)^2}\int_M |\nabla T(\varphi)^{\frac{N+2}{2}}|^2 dv + \int_M RT(\varphi)^{N+2}dv \\ &\geqslant \frac{4(N+1)}{(N+2)^2}\left(\frac{4(n-1)}{n-2}\int_M |\nabla T(\varphi)^{\frac{N+2}{2}}|^2 dv + \int_M RT(\varphi)^{N+2}dv\right) \quad (\text{since } R > 0) \\ &\geqslant \frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} \quad (\text{by the definition of } \mathcal{Y}_g) \\ &\geqslant \int_M |\sigma|^2 dv + \int_M |LW|^2 dv, \qquad (\text{by } (2.40))\end{aligned}$$

it follows from (2.41) that $\int_M \tau^2 T(\varphi)^{2N}dv \leqslant 0$, which is a contradiction.

Now let $m \in M$ s.t. $0 < T(\varphi)(m) < \varphi(m)$ (note that $T(\varphi) \in C^0_+$). By the continuity of T, we obtain that there exists $\delta = \delta(\varphi)$ s.t. for all $\theta \in B(\varphi,\delta) \cap \mathscr{C}$,

$$0 < T(\theta)(m) < \theta(m),$$

and hence from the fact that

$$-(S(\theta)(m)-\theta(m)) = \begin{cases} -(\min\{T(\theta)(m),a\}-\theta(m)) & \text{if } \frac{4(N+1)}{(N+2)^2}\mathcal{Y}_g\left(\int_M \theta^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} \leqslant 2\int_M |\sigma|^2 dv \\ \theta(m) & \text{otherwise,} \end{cases}$$

we conclude that $-(S(\theta)(m)-\theta(m)) > 0,\ \forall\theta\in B(\varphi,\delta)\cap\mathscr{C}$.

Hence, by the definition of half-continuity applied with $p(f) = -f(m)$ for all $f \in C^0$, we obtain that S is half-continuous at φ. The proof is completed. □

Remark 2.3.9. *From the proof above, a more precise assumption for Theorem 2.3.8 is that $\|d\tau\|_{L^p}\|\sigma\|_{L^2}^{\frac{(N-2)}{N+2}}$ is small enough, only depending on (M,g).*

2.3.3 A Sufficient Condition to the Existence of Solutions

We note that the main ingredient to prove the half-continuity of S in the two proofs above is the existence of $m \in M$ s.t. $T(\varphi)(m) < \varphi(m)$. This leads us to propose a sufficient condition for the existence of a solution to (3), which is much weaker than the concept of a global supersolution (see [Holst *et al.*, 2009] or [Maxwell, 2009]). We will begin with the notion of a local supersolution.

Definition 2.3.10. *Let data be given on M as specified in (4) and assume that (5) holds. We call $\theta \in L^\infty_+$ a local supersolution to (3) if for every positive function φ satisfying $\varphi \leqslant \theta$ and $\varphi = \theta$ somewhere, there exists $m \in M$ such that $T(\varphi)(m) \leqslant \varphi(m)$.*

Recall that $\theta \in L^\infty_+$ is called a global supersolution to (3) if for all $m \in M$,

$$\sup_{\substack{\varphi\leqslant\theta,\\ \varphi\in L^\infty_+}} T(\varphi)(m) \leqslant \theta(m).$$

It follows immediately that

Proposition 2.3.11. *A global supersolution is a local supersolution.*

Proof. Assume that θ is a global supersolution to (3). Let φ be an arbitrary positive function satisfying $\varphi \leqslant \theta$ and $\varphi = \theta$ somewhere. Taking $m \in M$ s.t. $\varphi(m) = \theta(m)$, by definition of a global supersolution, it is clear that

$$T(\varphi)(m) \leqslant \theta(m) = \varphi(m),$$

and hence θ is a local supersolution. □

Theorem 2.3.12. *Let data be given on M as specified in (4) and assume that (5) holds. Assume that $\theta \in L^\infty_+$ is a local supersolution to (3). Then (3) admits a solution.*

Proof. Let $\mathscr{C}$ be given by

$$\mathscr{C} = \left\{\varphi \in C^0 : 0 \leqslant \varphi \leqslant b\right\},$$

with b large enough s.t.

$$\sup_{\varphi\leqslant\theta} \|T(\varphi)\|_\infty < b.$$

Here recall that from the vector equation, the set $\left\{LW_\varphi : \varphi \leqslant \theta\right\}$ is uniformly bounded in L^∞ by $b_1 = b_1(M, g, \theta, \tau)$. Then, by Lemma 1.2.5, $\{T(\varphi) : \varphi \leqslant \theta\}$ is uniformly bounded (in L^∞) by $\max \theta_0$, where θ_0 is the unique solution to (1.2) associated to $w = b_1 + \|\sigma\|_\infty$, and hence b is well-defined.

We define

$$S(\varphi) = \begin{cases} T(\varphi) & \text{if } \varphi \leqslant \theta \\ 0 & \text{otherwise.} \end{cases} \tag{2.42}$$

By Proposition 2.2.3, T is a compact map from C^0 into C^0_+. Then S maps $\mathscr{C}$ into itself and $S(\mathscr{C})$ is precompact. Assume for the moment that the half-continuity of S is proven. By Corollary 2.3.5, S has a fixed point φ_0. We claim that $\varphi_0 \not\equiv 0$. Indeed, if is not true, then $0 = \varphi_0 = S(\varphi_0)$, hence $\varphi_0 = 0 \leqslant \theta$. We get from the definition of S that $S(\varphi_0) = T(\varphi_0) > 0$ which is a contradiction with $S(\varphi_0) = 0$. Since $\varphi_0 \not\equiv 0$, so is $S(\varphi_0)$, and the definition of S implies that $\varphi_0 = S(\varphi_0) = T(\varphi_0)$.

Now we prove the half-continuity of S on $\mathscr{C}$. Since T is continuous, so is S at φ satisfying $\varphi < \theta$ everywhere or $\varphi > \theta$ somewhere. The only remaining work is to show that S is half-continuous at φ s.t. $\varphi \leqslant \theta$ and $\varphi = \theta$ somewhere.

For such a φ, assume that there exists $m_0 \in M$ s.t.

$$T(\varphi)(m_0) < \varphi(m_0).$$

By the continuity of T, we can choose $\delta = \delta(\varphi) > 0$ s.t. for all $\eta \in B(\varphi, \delta) \cap \mathscr{C}$,

$$T(\eta)(m_0) < \eta(m_0),$$

and hence from the fact that

$$-(S(\eta)(m_0) - \eta(m_0)) = \begin{cases} -(T(\eta)(m_0) - \eta(m_0)) & \text{if } \eta \leqslant \theta \\ \eta(m_0) & \text{otherwise,} \end{cases}$$

we obtain that $-(S(\eta)(m_0) - \eta(m_0)) > 0$, $\forall \eta \in B(\varphi, \delta) \cap \mathscr{C}$. Now, by the definition of half-continuity applied with $p(f) = -f(m_0)$ for all $f \in C^0$, we conclude that S is half-continuous at φ.

It remains to study the case when $\varphi \leqslant T(\varphi)$. Since θ is a local supersolution, there exists m s.t. $T(\varphi)(m) \leqslant \varphi(m)$ and since $\varphi \leqslant T(\varphi)$, we have $T(\varphi)(m) = \varphi(m)$. Because the case $T(\varphi) \equiv \varphi$ is trivial [(φ, W_φ) is then a solution to (3)], we can assume that there exists $q \in M$ s.t. $T(\varphi)(q) > \varphi(q)$. Let $A, B > 0$ satisfying

$$A\varphi(m) - B\varphi(q) > 0. \tag{2.43}$$

Note that since $\varphi(m) = T(\varphi)(m) > 0$ ($T(\varphi) \in C^0_+$), such A, B exist. On the other hand, by the assumptions on q and m,

$$-A(T(\varphi)(m) - \varphi(m)) + B(T(\varphi)(q) - \varphi(q)) = -A.0 + B(T(\varphi)(q) - \varphi(q)) > 0. \tag{2.44}$$

By (2.43), (2.44) and the continuity of T, there exists $\delta_1 = \delta_1(\varphi) > 0$ small enough s.t. for all $\eta \in B(\varphi, \delta_1) \cap \mathscr{C}$

$$A\eta(m) - B\eta(q) > 0$$

and

$$-A\left(T(\eta)(m) - \eta(m)\right) + B\left(T(\eta)(q) - \eta(q)\right) > 0.$$

Therefore, by the fact that

$$-A(S(\eta)(m)-\eta(m))+B(S(\eta)(q)-\eta(q)) = \begin{cases} -A\left(T(\eta)(m) - \eta(m)\right) + B\left(T(\eta)(q) - \eta(q)\right) & \text{if } \eta \leqslant \theta \\ A\eta(m) - B\eta(q) & \text{otherwise,} \end{cases}$$

we obtain that $-A(S(\eta)(m) - \eta(m)) + B(S(\eta)(q) - \eta(q)) > 0$ for all $\eta \in B(\varphi, \delta_1) \cap \mathscr{C}$. Now, by the definition of half-continuity applied with $p(f) = -Af(m) + Bf(q)$ for all $f \in C^0$, we can conclude that S is half-continuous at φ. The proof is completed. □

A direct consequence of Theorem 2.3.12 is the following:

Corollary 2.3.13. *If $T(\varphi) \leqslant \varphi$ somewhere for every $\varphi \in L^\infty$ large enough, then (3) admits a solution.*

Chapter 3

Nonexistence and Nonuniqueness Results for Solutions to the Vacuum Einstein Conformal Constraint Equations

3.1 Introduction

The possibilities of nonexistence or nonuniqueness of solutions to (3) are studied by many authors in some particular situations for the scalar field case. For instance, as presented in Chapter 1, the Lichnerowicz equations seen as the CMC case of the system (3) can have zero or several solutions.

Conversely, nonexistence and nonuniqueness results for (3) in the vacuum case are fairly rare. While the first one relating to nonuniqueness results, addressed by Maxwell [Maxwell, 2011], shows that on the $n-$torus there exists a model leading to nonuniqueness of solutions to the vacuum system (3), the other, achieved by Isenberg [Isenberg et Ó Murchadha, 2004] and later strengthened in [Dahl *et al.*, 2012], [Gicquaud et Ngô, 2014], states that the system (3) with $\mathcal{Y}_g > 0$ and with $\sigma \equiv 0$ has no solution, provided $d\tau/\tau$ is small enough. This second assertion led Maxwell in 2009 to ask whether the non-zero assumption of τ is a necessary condition for the existence of solutions to system (3) with positive Yamabe invariants (see [Maxwell, 2009]).

In this chapter, we are interested in using the Leray-Schauder fixed point theorem for exhibiting a nonexistence and a nonuniqueness result for solutions to the vacuum system (3). The key in our treatment is an extension of Theorem 2.1.2 reproven in the previous chapter, which allows us to control the parameter α in the limit equation (2.2). More precisely, we have that

Theorem 3.1.1 (***Control of the parameter***)**.** *Let data be given on M as specified in (4) associated to the vacuum case, and assume that conditions (5) hold. If τ has constant sign and $\sigma \not\equiv 0$, then at least one of the following assertions is true*

(i) The conformal constraint equations (3) admit a solution (φ, W) with $\varphi > 0$. Furthermore,

the set of solutions $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$*, with* $p > n$*, is compact.*

(ii) There exists a nontrivial solution $W \in W^{2,p}$ *to the limit equation*

$$-\frac{1}{2}L^*LW = \sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{\tau}. \tag{3.1}$$

(iii) For all continuous functions $f > 0$ *or* $f \equiv R$ *if* $\mathcal{Y}_g > 0$*, the (modified) conformal constraint equations*

$$\frac{4(n-1)}{n-2}\Delta\varphi + f\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |LW|^2\varphi^{-N-1} \tag{3.2a}$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N d\tau \tag{3.2b}$$

have a (non-trivial) solution $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$*. Moreover if* $\mathcal{Y}_g > 0$*, then there exists a sequence* $\{t_i\}$ *converging to* 0 *s.t. the conformal constraint equations* (3) *associated to seed data* $(g, t_i\tau, \sigma)$ *have at least two solutions.*

Comparing with the original version of Dahl–Gicquaud–Humbert; i.e., Theorem 2.1.2, the price to pay to set the parameter $\alpha = 1$ in (2.2) is the addition of (*iii*). We will see that this assertion is necessary because of the following result.

Theorem 3.1.2 (***Nonexistence of solution***)**.** *Let data be given on* M *as specified in (4) associated to the vacuum case, and assume that conditions (5) hold. Assume further that* τ *has constant sign and there exists* $c > 0$ *s.t.* $\left|L\left(\frac{d\tau}{\tau}\right)\right| \leqslant c\left|\frac{d\tau}{\tau}\right|^2$*. Let* U *be a given open neighborhood of the critical set of* τ*. If* $\sigma \not\equiv 0$ *and* $\mathrm{supp}\{\sigma\} \subsetneq M \setminus U$*, then both of the conformal constraint equations* (3) *and the limit equation* (3.1) *associated to seed data* $(g, \tau^a, k\sigma)$ *admit no (nontrivial) solution, provided* a, k *are large enough.*

It is worth noting that [Dahl *et al.*, 2012, Proposition 1.6] provides the existence of seed data (M, g, τ, σ) satisfying the assumptions. In fact, our proof for Theorem 3.1.2 is an extension of arguments in [Dahl *et al.*, 2012, Proposition 1.6].

As direct consequences of Theorem 3.1.1 and 3.1.2, we also obtain the following results.

Corollary 3.1.3 (***An answer to Maxwell's question***)**.** *Let* (M, g, τ) *be given as in Theorem 3.1.2. If* $\mathcal{Y}_g > 0$*, then the conformal constraint equations* (3) *associated to* $(g, \tau^a, 0)$ *has a (nontrivial) solution for all* $a > 0$ *large enough.*

Corollary 3.1.4 (***Nonuniqueness of solutions***)**.** *Assume that* $(M, g, \tau, \sigma, a, k)$ *is given as in Theorem 3.1.2. If* $\mathcal{Y}_g > 0$*, there exists a sequence* $\{t_i\}$ *converging to* 0 *s.t. the conformal constraint equations* (3) *associated to data* $(g, t_i\tau^a, k\sigma)$ *has at least two solutions.*

Note that there is a small difference between Theorem 3.1.1 and related results claimed in the general introduction of this thesis. It is the addition of the assumption that $\sigma \not\equiv 0$. This is with the aim of simplifying the arguments and will be removed if $\mathcal{Y}_g < 0$ by Remark 3.2.2.

3.2 Proof of Theorem 3.1.1

In this section, we will use the Leray-Schauder fixed point theorem introduced in the previous chapter for obtaining another version of Theorem 2.1.2.

Before going further, it is worth making the following remark.

Remark 3.2.1. *In this section, we will study a modified version of* (1.2)*:*

$$\frac{4(n-1)}{n-2}\Delta\varphi + (tR + (1-t)f)\,\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{w^2}{\varphi^{N+1}} \tag{3.3}$$

where $t \in [0,1]$ is a parameter and $f > 0$ is a given continuous function. Since $(f,t) \in C^0_+ \times [0,1]$, standard facts stated above are still valid for this equation. For instance, given $\theta \in W^{2,p}_+$, similarly to the proof of Lemma 1.2.2, φ is a supersolution (resp. subsolution) to (3.3) if and only if $\hat\varphi = \theta^{-1}\varphi$ is a supersolution (resp. subsolution) to the following equation

$$\frac{4(n-1)}{n-2}\Delta_{\hat g}\hat\varphi + \left(t\hat R + (1-t)\hat f\right)\hat\varphi + \frac{n-1}{n}\hat\tau^2\hat\varphi^{N-1} = \frac{\hat w^2}{\hat\varphi^{N+1}},$$

where $\hat f = \theta^{-N+2}f$ and $(\hat g, \hat w, \hat\tau)$ is given as in Lemma 1.2.2. Thus, the conformal covariance still holds in our situation.

As a second example, we will see that existence and uniqueness of solutions given in Theorem 1.2.3 with $\mathcal{Y}_g < 0$ (i.e., $R < 0$ by Remark 1.2.4) are still true here. In fact, assume that $w \in L^{2p}\setminus\{0\}$. Then let $\phi_f > 0$ be the unique positive solution to

$$\frac{4(n-1)}{n-2}\Delta\varphi + R_f\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{w^2}{\varphi^{N+1}} \tag{3.4}$$

with $R_f = \sup_t(\max\{tR + (1-t)f\}) > 0$ (here existence and uniqueness of ϕ_f is proven similarly to Case 1 of Theorem 1.2.3). It is easy to see that ϕ_f is a subsolution to (3.3). On the other hand, let ϕ be the unique positive solution to the corresponding original Lichnerowicz equation (1.2). Since $f > 0$ and $t \in [0,1]$, ϕ is a supersolution to (3.3), and then the (modified) Lichnerowicz equation (3.3) admits a solution by the method of sub- and super-solution (note that since ϕ is also a supersolution to (3.4), $\phi \geqslant \phi_f$ by Lemma 1.2.5). Uniqueness of solutions follows by the same method as in [Maxwell, 2005, Proposition 4.4]. Similarly, it is not difficult to show that Lemma 1.2.5 remains valid for the (modified) Lichnerowicz equation by the same argument.

We are now ready to prove our main result.

Proof of Theorem 3.1.1. We divide the proof into three steps

Step 1. *Construction of a continuous compact operator:* Given any continuous function $f > 0$ or $f \equiv R$ if $\mathcal{Y}_g > 0$, we define the map $T_f : L^\infty \times [0,1] \to L^\infty$ as follows. For each $(\varphi, t) \in L^\infty \times [0,1]$, there exists a unique $W_\varphi \in W^{2,p}$ such that

$$-\frac{1}{2}L^*LW_\varphi = \frac{n-1}{n}\varphi^N d\tau, \tag{3.5}$$

and, by Remark 3.2.1, there is a unique $\phi_{\varphi,t} \in W^{2,p}_+$ satisfying

$$\frac{4(n-1)}{n-2}\Delta\phi_{\varphi,t} + [tR + (1-t)f]\,\phi_{\varphi,t} = -\frac{n-1}{n}t^{2N}\tau^2\phi_{\varphi,t}^{N-1} + |\sigma + LW_\varphi|^2\phi_{\varphi,t}^{-N-1}.$$

We define

$$T_f(\varphi, t) := t\phi_{\varphi,t}.$$

Following [Maxwell, 2009] and [Dahl *et al.*, 2012], the mapping $G:\ L^\infty \to C^1$ defined by $G(\varphi) = W_\varphi$, with W_φ uniquely determined by (3.5) is continuous and compact. Thus, to show that T_f is compact and continuous, it suffices to prove the continuity of $\hat{T}_f:\ C^1 \times [0,1] \to W^{2,p}_+$ defined by $\hat{T}_f(W,t) = \phi$, where

$$\frac{4(n-1)}{n-2}\Delta\phi + [tR + (1-t)f]\,\phi = -\frac{n-1}{n}t^{2N}\tau^2\phi^{(N-1)} + |\sigma + LW|^2\phi^{-N-1}. \tag{3.6}$$

We combine the techniques from [Dahl *et al.*, 2012, Lemma 2.3] and Proposition 2.2.6 to prove that $\hat{T}_f$ is continuous. Set $u = \ln \hat{T}_f(W,t)$. We have from the definition of $\hat{T}_f$ that

$$\frac{4(n-1)}{n-2}\left(\Delta u - |du|^2\right) + [tR + (1-t)f] = -\frac{n-1}{n}t^{2N}\tau^2 e^{(N-2)u} + |\sigma + LW|^2 e^{-(N+2)u}.$$

Next, we prove that $\ln \hat{T}_f$ is a C^1-map through the implicit function theorem. In fact, define $F : C^1 \times [0,1] \times W^{2,p} \to L^p$ by

$$F(W,t,u) = \frac{4(n-1)}{n-2}\left(\Delta u - |du|^2\right) + [tR + (1-t)f] + \frac{n-1}{n}t^{2N}\tau^2 e^{(N-2)u} - |\sigma + LW|^2 e^{-(N+2)u}.$$

It is clear that F is C^1 and, under our assumptions $u = \ln\left(\hat{T}_f(W,t)\right)$ is the unique solution to $F(W,t,u) = 0$. A standard computation shows that the Fréchet derivative of F w.r.t. u is given by

$$F_u(W,t)(v) = \frac{4(n-1)}{n-2}\left(\Delta v - \langle du, dv\rangle\right) + \frac{(n-1)(N-2)}{n}t^{2N}\tau^2 e^{(N-2)u}v + (N+2)|\sigma + LW|^2 e^{-(N+2)u}v.$$

We first note that $F_u \in C\left(C^1 \times [0,1], L(W^{2,p}, L^{2p})\right)$, where $L(W^{2,p}, L^{2p})$ denotes the Banach space of all linear continuous maps from $W^{2,p}$ into L^{2p}. In particular, setting $u_0 = \ln\left(\hat{T}_f(W,t)\right)$ we have

$$F_{u_0}(W,t)(v) = \frac{4(n-1)}{n-2}\left(\Delta v - \langle du_0, dv\rangle\right) + \left(\frac{(n-1)(N-2)}{n}t^{2N}\tau^2 e^{(N-2)u_0} + (N+2)|\sigma + LW|^2 e^{-(N+2)u_0}\right)v.$$

Since

$$\int_M |\sigma + LW|^2 e^{-(N+2)u_0}dv \geqslant e^{-(N+2)\max|u_0|}\int_M |\sigma + LW|^2 dv = e^{-(N+2)\max|u_0|}\left(\int_M |\sigma|^2 dv + \int_M |LW|^2 dv\right) > 0,$$

the non-negative term $\left(\frac{(n-1)(N-2)}{n}t^{2N}\tau^2 e^{(N-2)u_0} + (N+2)|\sigma + LW|^2 e^{-(N+2)u_0}\right)$ is not identically 0. Then we can conclude by the maximum principle that $F_{u_0}(W,t):\ W^{2,p} \to L^{2p}$ is an isomorphism (see [Gilbarg et Trudinger, 2001, Theorem 8.14]). The implicit function theorem then implies that $\ln \circ \hat{T}_f$ is a C^1-function in a neighborhood of (W,t), which proves our claim.

Step 2. *Application of the Leray-Schauder fixed point theorem:* We now set

$$K = \left\{ \varphi \in L^\infty |\ \exists t \in [0,1] \text{ such that } \varphi = T_f(\varphi, t) \right\}.$$

By the Leray-Schauder fixed point theorem, if K is bounded, then the system (3) associated to (g, τ, σ) admits a solution, which is our first assertion.

Assume from now on that K is unbounded. So there exists a sequence (φ_i, W_i, t_i) satisfying

$$\frac{4(n-1)}{n-2}\Delta\varphi_i + [t_i R + (1-t_i)f]\,\varphi_i = -\frac{n-1}{n}t_i^{2N}\tau^2\varphi_i^{N-1} + |\sigma + LW_i|^2\varphi_i^{-N-1} \quad (3.7a)$$

$$-\frac{1}{2}L^*LW_i = \frac{n-1}{n}t_i^N\varphi_i^N d\tau, \quad (3.7b)$$

with $\|\varphi_i\|_{L^\infty} \to +\infty$ (see Proposition 2.2.6). We need to discuss the following four possibilities.

- *Case 1. (after passing to a subsequence)* $t_i \to t_0 > 0$*:* Arguing similarly to the proof of Theorem 2.1.2 given in Section 2.2, we obtain existence of a nontrivial solution $V \in W^{2,p}$ to the limit equation

$$-\frac{1}{2}L^*LV = \sqrt{\frac{n-1}{n}}|LV|\frac{d\tau}{\tau},$$

  which is our second assertion.

- *Case 2. (after passing to a subsequence)* $t_i \to 0$*:* Note that Equations (3.7) say that the (modified) conformal constraint equations associated to the seed data $(g, t_i^N\tau, \sigma)$ have a solution (φ_i, W_i). To derive the last two assertions, we need to free τ of t_i in the seed data. Then, rather than considering $(g, t_i^N\tau, \sigma)$, by Remark 2.2.8 (with appearance of f and t_i cause no problem here), we can equivalently work on more suitable seed data, allowing to remove t_i from the mean curvature τ, and hence by straightforward calculations as seen below the sequence $\{t_i^n\varphi_i\}_{i\in\mathbb{N}}$ will naturally appear and play an important role in characterizing our case. In this context, there are three situations arising depending on whether (after passing to subsequence) $t_i^n\|\varphi_i\|_{L^\infty}$ converges to $+\infty$, 0 or a positive constant. We will address each of them.

  In the first situation; i.e., $t_i^n\|\varphi_i\|_{L^\infty} \to +\infty$, by Remark 2.2.8, the system (3.7) may be rewritten as

$$\frac{4(n-1)}{n-2}\Delta\overline{\varphi}_i + [t_i R + (1-t_i)f]\,\overline{\varphi}_i = -\frac{n-1}{n}\tau^2\overline{\varphi}_i^{N-1} + \left|t_i^{\frac{n(N+2)}{2}}\sigma + L\overline{W}_i\right|^2\overline{\varphi}_i^{-N-1} \quad (3.8a)$$

$$-\frac{1}{2}L^*L\overline{W}_i = \frac{n-1}{n}\overline{\varphi}_i^N d\tau, \quad (3.8b)$$

  where $(\overline{\varphi}_i, \overline{W}_i) = \left(t_i^n\varphi_i, t_i^{\frac{n(N+2)}{2}}W_i\right)$ and $\|\overline{\varphi}_i\|_{L^\infty} = t_i^n\|\varphi_i\|_{L^\infty} \to \infty$. Again, taking $i \to \infty$ we argue similarly to Case 1 and obtain that there exists a nontrivial solution $\overline{W}_\infty \in W^{2,p}$ to the limit equation (3.1) as stated in (*ii*).

The next situation; i.e., $t_i^n\|\varphi_i\|_{L^\infty} \to 0$, cannot happen. In fact, also by Remark 2.2.8 the system (3.7) may be rewritten as

$$\begin{aligned} \frac{4(n-1)}{n-2}\Delta\widehat{\varphi}_i + [t_iR + (1-t_i)f]\widehat{\varphi}_i &= -\frac{n-1}{n}t_i^{2N}\gamma_i^{N-2}\tau^2\widehat{\varphi}_i^{N-1} + \left|\gamma_i^{-\frac{N+2}{2}}\sigma + L\widehat{W}_i\right|^2\widehat{\varphi}_i^{-N-1} && (3.9a)\\ -\frac{1}{2}L^*L\widehat{W}_i &= \frac{n-1}{n}t_i^N\gamma_i^{\frac{N-2}{2}}\widehat{\varphi}_i^N d\tau, && (3.9b)\end{aligned}$$

where $\gamma_i = \|\varphi_i\|_{L^\infty}$ and $(\widehat{\varphi}_i, \widehat{W}_i) = (\gamma_i^{-1}\varphi_i, \gamma_i^{-\frac{N+2}{2}}W_i)$. We recall that by Remark 1.2.4, R is assumed to be positive if $\mathcal{Y}_g > 0$, and then we may assume that $f > 0$ without loss of generality. At any maximum point m_i of $\widehat{\varphi}_i$ (i.e., $\widehat{\varphi}_i(m_i) = 1$), we have from (3.9a) that

$$\left([t_iR + (1-t_i)f] + \frac{n-1}{n}t_i^{2N}\gamma_i^{N-2}\tau^2\right)(m_i) \leqslant \left|\gamma_i^{-\frac{N+2}{2}}\sigma + L\widehat{W}_i\right|^2(m_i).$$

However, since $\|\widehat{\varphi}_i\|_{L^\infty} = 1$ and $t_i^n\gamma_i \to 0$, we obtain from the vector equation (3.9b) that $\|L\widehat{W}_i\|_{L^\infty} \to 0$, and then by the fact that $t_i \to 0$ and $\gamma_i \to +\infty$, taking $i \to \infty$ we conclude from the previous inequality that $0 < \min f \leqslant 0$, which is a contradiction as claimed.

For the last situation; i.e., $t_i^n\|\varphi_i\|_{L^\infty} \to c$ for some $c > 0$, by Remark 2.2.8, we again obtain the system (3.8) where the condition $\|\overline{\varphi}_i\|_{L^\infty} \to +\infty$ is replaced by $\|\overline{\varphi}_i\|_{L^\infty} \to c$. It follows from (3.8b) that (after passing to a subsequence) $\overline{W}_i$ converges to $\overline{W}_0$ in C^1. Regarding Remark 1.2.4, we may assume that $f > 0$. If $L\overline{W}_0 \equiv 0$, at any maximum point m_i of $\overline{\varphi}_i$ we have by (3.8a) that

$$0 < \left([t_iR + (1-t_i)f]\,\overline{\varphi}_i^{N+2} + \frac{n-1}{n}\tau^2\overline{\varphi}_i^{2N}\right)(m_i) \leqslant \left|t_i^{\frac{n(N+2)}{2}}\sigma + L\overline{W}_i\right|^2(m_i) \to 0.$$

This is a contradiction since

$$\left([t_iR + (1-t_i)f]\,\overline{\varphi}_i^{N+2} + \frac{n-1}{n}\tau^2\overline{\varphi}_i^{2N}\right)(m_i) \to f(m_0)c^{N+2} + \frac{n-1}{n}c^{2N}\tau^2(m_0) > 0,$$

where by compactness of M (after passing to a subsequence) m_i converges to $m_0 \in M$. Thus, we obtain $L\overline{W}_0 \not\equiv 0$. Now we can let $\overline{\varphi}_0$ be the unique positive solution to the equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + f\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |L\overline{W}_0|^2\varphi^{-N-1}.$$

(Here since $f > 0$, existence and uniqueness of $\overline{\varphi}_0$ is proven similarly to Case 1 of Theorem 1.2.3). To show that $(\overline{\varphi}_0, \overline{W}_0)$ is a (nontrivial) solution to system (3.2), which is the first statement of our last assertion, it suffices to show that $\overline{\varphi}_i \to \overline{\varphi}_0$ in L^∞. In fact, since $L\overline{W}_0 \not\equiv 0$, arguing similarly to the continuity of $\hat{T}_f$ in Step 1, we obtain that the map $\widetilde{T}_f : U_{\overline{W}_0} \times [0,1] \to W_+^{2,p}$ defined by $\widetilde{T}_f(w,t) = \varphi$ is continuous, where $U_{\overline{W}_0}$ is any small enough open neighborhood of $|L\overline{W}_0|$ in L^∞ and φ is the unique positive solution to the equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + [tR + (1-t)f]\,\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + w^2\varphi^{-N-1}.$$

Combining this and the fact that $\left(t_i, \left|t_i^{\frac{n(N+2)}{2}}\sigma + L\overline{W}_i\right|\right) \to \left(0, |L\overline{W}_0|\right)$ we obtain $\overline{\varphi}_i \to \overline{\varphi}_0$ as claimed.

To complete our proof, the remaining work is to treat nonuniqueness results for the conformal constraint equations with positive Yamabe invariants.

Step 3. *Nonuniqueness of solutions:* Assume that $\mathcal{Y}_g > 0$. If neither (i) nor (ii) is true, taking $f \equiv R$, arguments above then tell us that there exists a sequence $\{t_i\}$ converging to 0 s.t. the system (3) associated to $(g, t_i^N \tau, \sigma)$ has a solution (φ_i, W_i) satisfying $\|\varphi_i\|_{L^\infty} \to \infty$. On the other hand, we know that provided $\delta > 0$ is small enough, the system (3) associated to $(g, \delta\tau, \sigma)$ admits a solution $(\varphi_\delta, W_\delta)$ such that $\|\varphi_\delta\|_{L^\infty} \leqslant c_1$ for some constant $c_1 > 0$ independent of δ (see Remark 2.3.9 or [Gicquaud et Ngô, 2014, Theorem 2.1]). This completes the proof of Theorem 3.1.1. □

Remark 3.2.2. *If $\mathcal{Y}_g < 0$, we can omit the assumption $\sigma \not\equiv 0$ in Theorem 3.1.1. In fact, let $\{\sigma_i\}$ be a sequence of non-zero transverse-traceless tensors converging to 0. Applying Theorem 3.1.1 for $\sigma = \sigma_i$, if neither assertion (ii) nor (iii) is satisfied for all $i \in \mathbb{N}$, the system* (3) *associated to $\sigma = \sigma_i$ has a solution (φ_i, W_i). Moreover, these solutions must be uniformly bounded since we assumed that the assertion (ii) is not satisfied. Note that by Case 3 of Theorem 1.2.3 and Lemma 1.2.5 we have that $\varphi_i \geqslant \min \varphi_0 > 0$, where φ_0 is the unique positive solution to the Yamabe equation*

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1}.$$

Thus, taking $i \to \infty$, we obtain our claim.

3.3 Applications of Theorem 3.1.1

In this section, we show a nonexistence and nonuniqueness result and answer a question raised in [Maxwell, 2009] (see the middle paragraph of page 630) as stated in the beginning of this chapter. For convenience, we will repeat the statements and give the corresponding proofs. We first exhibit a class of seed data such that the corresponding equations (3) and (3.1) have no (non-trivial) solution.

Theorem 3.3.1. ***(Nonexistence of solution)*** *Let data be given on M as specified in* (4) *and assume that conditions* (5) *hold. Furthermore, assume that there exists $c > 0$ s.t. $\left|L\left(\frac{d\tau}{\tau}\right)\right| \leqslant 2c\left|\frac{d\tau}{\tau}\right|^2$. Let V be a given open neighborhood of the critical set of τ. If $\sigma \not\equiv 0$ and $supp\{\sigma\} \subsetneq M \setminus V$, then both the conformal constraint equations* (3) *and the limit equation* (3.1) *associated to the seed data $(g, \tau^a, \frac{\sigma}{\epsilon a})$ have no solution, provided $a^{-1}, \epsilon a > 0$ are small enough.*

Examples where the assumptions of this theorem hold are given in [Dahl *et al.*, 2012]. Let us sketch briefly their construction. Let M be the sphere $\mathbb{S}^n$ endowed with the round metric. Choose $\tau = \exp(x_1)$ so that $(d\tau/\tau)^\sharp$ is a conformal Killing vector field for the round metric Ω on $\mathbb{S}^n$. The critical set of τ then consists of the points $(\pm 1, 0, \ldots, 0)$. Let V be an arbitrary neighborhood of these points such that $\mathbb{S}^n \setminus V$ has non-empty interior. By a result of [Beig *et al.*, 2005], we can deform the metric Ω on $\mathbb{S}^n \setminus V$ to a new metric g so that g has no conformal Killing vector. The condition $\left|L\left(\frac{d\tau}{\tau}\right)\right| \leqslant 2c\left|\frac{d\tau}{\tau}\right|^2$ is then readily checked. Non-trivial TT-tensors with arbitrarily small support were constructed in [Delay,]. His construction shows that there exists $\sigma \not\equiv 0$ whose support is contained in $\mathbb{S}^n \setminus V$.

Proof of Theorem 3.3.1. We argue by contradiction. Assume that for each (a, ϵ) s.t. $a^{-1}, \epsilon a > 0$ are small enough, there exists $(\varphi_{\epsilon,a}, W_{\epsilon,a})$ satisfying the conformal constraint equations

$$\frac{4(n-1)}{n-2}\Delta\varphi_{\epsilon,a} + R\varphi_{\epsilon,a} = -\frac{n-1}{n}\tau^{2a}\varphi_{\epsilon,a}^{N-1} + \left|\frac{\sigma}{\epsilon a} + LW_{\epsilon,a}\right|^2 \varphi_{\epsilon,a}^{-N-1}, \tag{3.10a}$$

$$-\frac{1}{2}L^* LW_{\epsilon,a} = \frac{n-1}{n}\varphi_{\epsilon,a}^N d\tau^a. \tag{3.10b}$$

We will use the rescaling idea of Dahl–Gicquaud–Humbert [Dahl *et al.*, 2012] to show that such existence yields a contradiction. In fact, we rescale $\varphi_{\epsilon,a}$, $W_{\epsilon,a}$ as follows

$$\widetilde{\varphi}_{\epsilon,a} = \epsilon^{\frac{1}{N}}\varphi_{\epsilon,a}, \quad \widetilde{W}_{\epsilon,a} = \epsilon W_{\epsilon,a}.$$

The system (3.10) may be written as

$$\epsilon^{\frac{2}{n}}\widetilde{\varphi}_{\epsilon,a}^{N+1}\left(\frac{4(n-1)}{n-2}\Delta\widetilde{\varphi}_{\epsilon,a} + R\widetilde{\varphi}_{\epsilon,a}\right) = -\frac{n-1}{n}\tau^{2a}\widetilde{\varphi}_{\epsilon,a}^{2N} + \left|\frac{\sigma}{a} + L\widetilde{W}_{\epsilon,a}\right|^2, \tag{3.11a}$$

$$-\frac{1}{2}L^* L\widetilde{W}_{\epsilon,a} = \frac{n-1}{n}\widetilde{\varphi}_{\epsilon,a}^N d\tau^a. \tag{3.11b}$$

We divide our proof into two cases.

Case 1. $\lim_{\epsilon\to 0}\left\|\widetilde{\varphi}_{\epsilon,a}\right\|_{L^\infty} < \infty$*:* Arguing as in the proof of Theorem 3.1.1, taking $\epsilon \to 0$ we obtain that there exists $W_a \in W^{2,p}$ satisfying

$$\begin{aligned}-\frac{1}{2}L^* LW_a &= \sqrt{\frac{n-1}{n}}\left|\frac{\sigma}{a} + LW_a\right|\frac{d\tau^a}{\tau^a} \\ &= \sqrt{\frac{n-1}{n}}\left|\sigma + aLW_a\right|\frac{d\tau}{\tau}.\end{aligned} \tag{3.12}$$

However, (3.12) cannot happen for all $a > 0$ large enough by [Dahl *et al.*, 2012, Proposition 1.6]. In fact, take the scalar product of this equation with $d\tau/\tau$ and integrate. It follows that

$$\begin{aligned}\sqrt{\frac{n-1}{n}}\int_M |\sigma + aLW_a|\left|\frac{d\tau}{\tau}\right|^2 dv &= -\frac{1}{2}\int_M \langle LW_a, L(d\tau/\tau)\rangle dv \\ &\leqslant c\int_M \left|\frac{d\tau}{\tau}\right|^2 |LW_a| dv \quad \text{(by our assumption).}\end{aligned} \tag{3.13}$$

Combining this with the fact that $|\sigma + aLW_a| \geqslant a|LW_a| - |\sigma|$, we conclude that for $c_1 = \sqrt{\frac{n}{n-1}}c$

$$(a - c_1)\int_M \left|\frac{d\tau}{\tau}\right|^2 |LW_a| dv \leqslant \int_M |\sigma|\left|\frac{d\tau}{\tau}\right|^2 dv.$$

Since the right-hand side of the inequality above is bounded, we must have

$$\lim_{a\to\infty}\int_M \left|\frac{d\tau}{\tau}\right|^2 |LW_a| dv = 0. \tag{3.14}$$

It then follows from (3.13) that

$$\lim_{a\to\infty}\int_M |\sigma + aLW_a|\left|\frac{d\tau}{\tau}\right|^2 dv = 0.$$

Since $\left|\frac{d\tau}{\tau}\right| \geqslant \delta$ on $M \setminus V$ for some $\delta > 0$ independent of a, we then have by the previous inequality that

$$\lim_{a\to\infty}\int_{M\setminus V} |\sigma + aLW_a| dv = 0. \tag{3.15}$$

On the other hand, since $\operatorname{supp}\{\sigma\} \subsetneq M \setminus V$, we get that

$$\left|\int_M \langle \sigma, \sigma + aLW_a\rangle dv\right| \leqslant \|\sigma\|_{L^\infty}\int_{M\setminus V} |\sigma + aLW_a| dv.$$

Together with (3.15), this shows that

$$\lim_{a\to\infty}\int_M \langle \sigma, aLW_a\rangle dv = -\int_M |\sigma|^2 dv. \tag{3.16}$$

However, since σ is divergence-free, we must have

$$\int_M \langle \sigma, aLW_a\rangle dv = 0$$

for all $a > 0$, which contradicts (3.16).

Case 2. $\lim_{\epsilon\to 0}\left\|\widetilde{\varphi}_{\epsilon,a}\right\|_{L^\infty} = +\infty$*:* Set $\gamma_{\epsilon,a} = \left\|\widetilde{\varphi}_{\epsilon,a}\right\|_{L^\infty}$; we rescale $\widetilde{\varphi}_{\epsilon,a}$, $\widetilde{W}_{\epsilon,a}$, $\widetilde{\sigma}_{\epsilon,a}$ again

$$\widehat{\varphi}_{\epsilon,a} = \gamma_{\epsilon,a}^{-1}\widetilde{\varphi}_{\epsilon,a}, \quad \widehat{W}_{\epsilon,a} = \gamma_{\epsilon,a}^{-N}\widetilde{W}_{\epsilon,a}, \quad \text{and} \quad \widehat{\sigma}_{\epsilon,a} = \gamma_{\epsilon,a}^{-N}\sigma.$$

The system (3.11) may be rewritten as

$$\epsilon^{\frac{2}{n}}\gamma_{\epsilon,a}^{-(N-2)}\widehat{\varphi}_{\epsilon,a}^{N+1}\left(\frac{4(n-1)}{n-2}\Delta\widehat{\varphi}_{\epsilon,a} + R\widehat{\varphi}_{\epsilon,a}\right) = -\frac{n-1}{n}\tau^{2a}\widehat{\varphi}_{\epsilon,a}^{2N} + \left|\frac{\widehat{\sigma}}{a} + L\widehat{W}_{\epsilon,a}\right|^2, \tag{3.17a}$$

$$-\frac{1}{2}L^*L\widehat{W}_{\epsilon,a} = \frac{n-1}{n}\widehat{\varphi}_{\epsilon,a}^{N}d\tau^a. \tag{3.17b}$$

Arguing as in the proof of Theorem 3.1.1, and taking $\epsilon \to 0$ we again obtain that there exists a nontrivial solution $W_a \in W^{2,p}$ satisfying the limit equation

$$-\frac{1}{2}L^*LW_a = \sqrt{\frac{n-1}{n}}|LW_a|\frac{d\tau^a}{\tau^a} = a\sqrt{\frac{n-1}{n}}|LW_a|\frac{d\tau}{\tau}. \tag{3.18}$$

Our treatment for this limit equation is also similar to the previous case. In fact, take the scalar product of this equation with $d\tau/\tau$ and integrate. It follows that

$$\begin{aligned} a\sqrt{\frac{n-1}{n}}\int_M |LW_a|\left|\frac{d\tau}{\tau}\right|^2 dv &= -\frac{1}{2}\int_M \langle LW_a, L(d\tau/\tau)\rangle dv \\ &\leqslant c\int_M |LW_a|\left|\frac{d\tau}{\tau}\right|^2 dv \ \ \text{(by our assumption)}. \end{aligned} \tag{3.19}$$

Then assuming $a > \sqrt{\frac{n}{n-1}}c$, we obtain that $\int_M |LW_a| \left|\frac{d\tau}{\tau}\right|^2 dv = 0$, and hence $|LW_a| \left|\frac{d\tau}{\tau}\right| \equiv 0$. Thus, we obtain from (3.18) that $W_a \equiv 0$, since (M, g) has no conformal Killing vector field. This is a contradiction with the fact that W_a is nontrivial.

Since Case 2 coincides with the situation of nonexistence of solution to the limit equation (3.1). □

As direct consequences of Theorem 3.1.1 and 3.3.1, we have the following results.

Corollary 3.3.2. *(**An answer to Maxwell's question**) Let (M, g, τ) be given as in Theorem 3.3.1. If $\mathcal{Y}_g > 0$, then the conformal constraint equations (3) associated to $(g, \tau^a, 0)$ have a (nontrivial) solution for all $a > 0$ large enough.*

Proof. Taking $f \equiv R$ in the proof of Theorem 3.1.1, we have by Theorem 3.3.1 that for all $a^{-1}, \epsilon a > 0$ small enough, seed data $(g, \tau^a, \frac{\sigma}{\epsilon a})$ satisfies neither (i) nor (ii) in Theorem 3.1.1, provided σ is given as in Theorem 3.3.1. Thus, our corollary is proven by the first statement in the assertion (iii) of Theorem 3.1.1. The proof is completed. □

Corollary 3.3.3. *(**Nonuniqueness of solutions**) Assume that $(M, g, \tau, \sigma, a, \epsilon)$ is given as in Theorem 3.3.1. If $\mathcal{Y}_g > 0$, then there exists a sequence $\{t_i\}$ converging to 0 s.t. the conformal constraint equations (3) associated to $(g, t_i\tau^a, \frac{\sigma}{\epsilon a})$ have at least two solutions.*

Proof. The same arguments as in Corollary 3.3.2 works here. More precisely, the only difference from the previous corollary is that we use the second conclusion in the assertion (iii) of Theorem 3.1.1 instead of the first, and then the corollary follows. □

Chapter 4

Solutions to the Einstein-Scalar Field Constraint Equations with a Small TT-Tensor

4.1 Introduction

In this chapter and in the next one, we will treat the Einstein constraint equations in the scalar field case. The most natural approach to (3) is to extend known results in the vacuum case to the scalar field one. However, the difficulty is that the methods used in [Holst *et al.*, 2009], [Maxwell, 2009] cannot work any longer since the Lichnerowicz equation may admit no or multiple solutions when $\mathcal{B}_{\tau,\psi}$ has arbitrary sign as shown in Chapter 2. This seems to tell us to seek new methods. So far, no studies have gone beyond the near-CMC recently presented by Premoselli [Premoselli, 2014] under an assumption that the generalized conformal Laplacian

$$L_{g,\psi} : \phi \mapsto \frac{4(n-1)}{n-2}\Delta\phi + \mathcal{R}_\psi\phi \tag{4.1}$$

is a coercive operator, meaning that there exists a constant $c > 0$ such that

$$\forall \phi \in W^{1,2}, \int_M \phi L_{g,\psi}\phi d\mu^g \geqslant c\, \|\phi\|^2_{W^{1,2}}.$$

This chapter is a joint work with Romain Gicquaud. In the next sections, we will show that the assumption above also plays a role analogous to one that the metric g has positive Yamabe invariant in [Holst *et al.*, 2008, Holst *et al.*, 2009, Maxwell, 2009] for constructing solutions to (3) with freely specified mean curvature, and hence gives the first existence result for solutions to the scalar field system (3) in the far-from-CMC cases.

The outline of this chapter is as follows. We will present the extension of the result of [Gicquaud et Ngô, 2014] in Section 4.2; see Theorem 4.2.1. Next, we address the much more difficult extension of the method of [Holst *et al.*, 2009] in Section 4.3; see Theorem 4.3.1. In the course of the proof, we prove Theorem 4.3.2 which shows existence of solutions to the Lichnerowicz equation in our context.

4.2 An implicit function argument

In this section, we show that the method introduced in [Gicquaud et Ngô, 2014] can be straightforwardly generalized to the system (3).

Theorem 4.2.1. *Let (M, g) be a compact Riemannian manifold, and let $\tau \in W^{1,p}$, $\psi \in W^{1,p}$, $\widetilde{\pi} \in L^p$ and $\widetilde{\sigma} \in L^p$ be given. Assume further that the operator $L_{g,\psi}$ defined in (4.1) is coercive and that (M, g) has no non-zero conformal Killing vector field. There exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the system (3) with*

$$\sigma \equiv \epsilon\widetilde{\sigma}, \quad \pi \equiv \epsilon\widetilde{\pi}$$

has a solution $(\phi, W) \in W^{2,\frac{p}{2}} \times W^{2,\frac{p}{2}}$ with $\phi > 0$.

As in the article [Gicquaud et Ngô, 2014], we divide the proof into several steps:

Step 0. *There exists a unique solution $\widetilde{W}_0 \in W^{2,\frac{p}{2}}$ to*

$$-\frac{1}{2}\mathbb{L}^*\mathbb{L}W = \widetilde{\pi}d\psi. \tag{4.2}$$

Proof. The argument is standard; see e.g. [Maxwell, 2009, Proposition 5]. Note that $\widetilde{\pi}d\psi \in L^{\frac{p}{2}}$. The operator

$$-\frac{1}{2}\mathbb{L}^*\mathbb{L} : W^{2,\frac{p}{2}} \to L^{\frac{p}{2}}$$

is Fredholm with zero index. Its kernel is, by a simple integration by parts argument, the set of conformal Killing vector fields which is reduced to $\{0\}$ by assumption. Hence $-\frac{1}{2}\mathbb{L}^*\mathbb{L}$ is an isomorphism. □

Step 1. *There exists a unique solution $\widetilde{\phi}_0 \in W^{2,\frac{p}{2}}$ to the following equation:*

$$\frac{4(n-1)}{n-2}\Delta\phi + \mathcal{R}_\psi\phi = \frac{\left|\widetilde{\sigma} + \mathbb{L}\widetilde{W}_0\right|^2 + \widetilde{\pi}^2}{\phi^{N+1}}. \tag{4.3}$$

Proof. We set

$$\widetilde{A} := \left|\widetilde{\sigma} + \mathbb{L}\widetilde{W}_0\right|^2 + \widetilde{\pi}^2$$

for convenience. Since $\widetilde{W}_0 \in W^{2,\frac{p}{2}}$, $\mathbb{L}\widetilde{W}_0 \in W^{1,\frac{p}{2}} \hookrightarrow L^p$. Indeed, from the Sobolev injection, $W^{1,\frac{p}{2}} \hookrightarrow L^q$, where

$$q = \frac{np}{(2n-p)^+} > p$$

(here $q = +\infty$ if $p \geqslant 2n$). It follows that $\widetilde{A} \in L^{\frac{p}{2}}$. We first prove that there exists a unique positive solution $\overline{\varphi}$ to

$$\frac{4(n-1)}{n-2}\Delta\phi + \mathcal{R}_\psi\phi = \widetilde{A}. \tag{4.4}$$

We remark that, integrating the right-hand side, we get

$$\begin{aligned}\int_M \widetilde{A}d\mu^g &= \int_M \left(|\widetilde{\sigma}|^2 + \widetilde{\pi}^2\right) d\mu^g + \int_M |\mathbb{L}W|^2 d\mu^g \\ &\geqslant \int_M \left(|\widetilde{\sigma}|^2 + \widetilde{\pi}^2\right) d\mu^g \\ &> 0.\end{aligned}$$

We rely on the Lax-Milgram theorem. Since $L_{g,\psi}$ is coercive, there exists a unique weak solution $\overline{\varphi}$ to (4.4) which is uniquely characterized by

$$\int_M \left(\frac{2(n-1)}{n-2} |d\overline{\varphi}|^2 + \frac{\mathcal{R}_\psi}{2} \overline{\varphi}^2 - \widetilde{A}\overline{\varphi} \right) d\mu^g = \min_{\phi \in W^{1,2}} F(\phi),$$

where

$$F(\phi) := \int_M \left(\frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\mathcal{R}_\psi}{2} \phi^2 - \widetilde{A}\phi \right) d\mu^g.$$

Since $\widetilde{A} \geqslant 0$, we have $F(|\phi|) \leqslant F(\phi)$ for any $\phi \in W^{1,2}$. As a consequence $\overline{\varphi}$ being the unique minimizer of F, $\overline{\varphi} \geqslant 0$. By elliptic regularity, we have that $\overline{\varphi} \in W^{2,\frac{p}{2}}$. In particular, $\overline{\varphi}$ is continuous. It can be argued by contradiction that $\overline{\varphi} > 0$. Indeed, if the set $\Omega = \{\overline{\varphi} = 0\}$ were not empty, it would follow from the Harnack inequality we borrow from [Trudinger, 1967, Theorems 1.1 and 5.1] applied to $u = \overline{\varphi}$ and $f \equiv 0$ in a ball B_R centered at a boundary point of Ω that $\overline{\varphi} \equiv 0$ on B_R, which is a contradiction.

Setting $a := \min_M \overline{\varphi}$, $b := \max_M \overline{\varphi}$, one can readily check that the function

$$\overline{\varphi}_+ := a^{-\frac{N+1}{N}} \overline{\varphi} \left(\text{resp. } \overline{\varphi}_- := b^{-\frac{N+1}{N}} \overline{\varphi} \right),$$

is a supersolution (resp. a subsolution) for Equation (4.3). Existence of a solution to (4.3) follows then from the standard sub- and supersolution method; see e.g. [Gicquaud et Huneau, 2014, Lemma 3.4] or [Maxwell, 2005]. Uniqueness of $\widetilde{\phi}_0$ is also classical; see [Dahl *et al.*, 2012]. However, here we can simply remark that the functional

$$G(\phi) := \int_M \left(\frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\mathcal{R}_\psi}{2} \phi^2 + \frac{1}{N} \frac{\widetilde{A}}{\phi^N} \right) d\mu^g$$

is strictly convex on the set of positive H^1-functions (i.e., so that there exists $\epsilon > 0$ such that $\phi \geqslant \epsilon$ a.e.) Its critical points being exactly the solutions to (4.3), we conclude that the solution to (4.3) is unique. This idea will be developed further in Section 4.3.1. □

Step 2. *There exists $\epsilon > 0$ and a C^1-map*

$$\begin{array}{ccc} [0, \epsilon) & \to & W^{2,\frac{p}{2}} \times W^{2,\frac{p}{2}} \\ \lambda & \mapsto & (\widetilde{\phi}_\lambda, \widetilde{W}_\lambda) \end{array}$$

such that

- *$\widetilde{\phi}_\lambda$ and $\widetilde{W}_\lambda$ solve*

$$\frac{4(n-1)}{n-2} \Delta \widetilde{\phi}_\lambda + \mathcal{R}_\psi \widetilde{\phi}_\lambda = \lambda^2 \mathcal{B}_{\tau,\psi} \widetilde{\phi}_\lambda^{N-1} + \frac{\left|\widetilde{\sigma} + \mathbb{L}\widetilde{W}_\lambda\right|^2 + \widetilde{\pi}^2}{\widetilde{\phi}_\lambda^{N+1}}, \tag{4.5a}$$

$$-\frac{1}{2} \mathbb{L}^* \mathbb{L} \widetilde{W}_\lambda = \frac{n-1}{n} \lambda \widetilde{\phi}_\lambda^N d\tau + \widetilde{\pi} d\psi. \tag{4.5b}$$

- *$\widetilde{\phi}_\lambda \to \widetilde{\phi}_0$ and $\widetilde{W}_\lambda \to \widetilde{W}_0$ when $\lambda \to 0$, where $\widetilde{W}_0$ and $\widetilde{\phi}_0$ are as defined in Steps 0 and 1.*

Note that Equations (4.5) interpolate between the original conformal constraint equations (3) when $\lambda = 1$ and Equations (4.3)-(4.2) when $\lambda = 0$.

Proof. The proof is via the implicit function theorem. Let $\Phi : \mathbb{R} \times W^{2,\frac{p}{2}} \times W^{2,\frac{p}{2}} \to L^{\frac{p}{2}} \times L^{\frac{p}{2}}$ be the following operator:

$$\Phi_\lambda : \begin{pmatrix} \widetilde{\phi} \\ \widetilde{W} \end{pmatrix} \mapsto \begin{pmatrix} \frac{4(n-1)}{n-2} \Delta \widetilde{\phi} + \mathcal{R}_\psi \widetilde{\phi} - \lambda^2 \mathcal{B}_{\tau,\psi} \widetilde{\phi}^{N-1} - \frac{\widetilde{A}}{\widetilde{\phi}^{N+1}} \\ -\frac{1}{2} \mathbb{L}^* \mathbb{L} \widetilde{W} - \frac{n-1}{n} \lambda \widetilde{\phi}^N d\tau - \widetilde{\pi} d\psi \end{pmatrix}.$$

Its differential with respect to the variables $(\widetilde{\phi}, \widetilde{W})$ at $(\lambda = 0, \widetilde{\phi}_0, \widetilde{W}_0)$ is given by the following block upper triangular matrix:

$$D\Phi_{\lambda=0}(\widetilde{\phi}_0, \widetilde{W}_0) = \begin{pmatrix} \frac{4(n-1)}{n-2} \Delta + \mathcal{R}_\psi + (N+1) \frac{\left|\widetilde{\sigma} + \mathbb{L} \widetilde{W}_0\right|^2 + \widetilde{\pi}^2}{\widetilde{\phi}_0^{N+2}} & -\frac{2}{\widetilde{\phi}_0^{N+1}} \left\langle \widetilde{\sigma} + \mathbb{L} \widetilde{W}_0, \mathbb{L} \cdot \right\rangle \\ 0 & -\frac{1}{2} \mathbb{L}^* \mathbb{L} \cdot \end{pmatrix}.$$

Each diagonal block is Fredholm with zero index and has, by assumption, a trivial kernel. This proves that $D\Phi_{\lambda=0}(\widetilde{\phi}_0, \widetilde{W}_0)$ is invertible. The existence of the curve of solutions to (4.5) on some interval $[0, \epsilon)$ is then guaranteed by the implicit function theorem. □

The last step is a straightforward calculation.

Step 3. *Let $(\widetilde{\phi}_\lambda, \widetilde{W}_\lambda)$ be as in Step 2. Setting*

$$\begin{cases} \phi_\lambda := \lambda^{\frac{2}{N-2}} \widetilde{\phi}_\lambda, \\ W_\lambda := \lambda^{\frac{N+2}{N-2}} \widetilde{W}_\lambda, \\ \sigma_\lambda := \lambda^{\frac{N+2}{N-2}} \widetilde{\sigma}, \\ \pi_\lambda := \lambda^{\frac{N+2}{N-2}} \widetilde{\pi}, \end{cases}$$

then $(\phi_\lambda, W_\lambda)$ solves the system (3) with $\sigma = \sigma_\lambda$ and $\pi = \pi_\lambda$.

The proof of Theorem 4.2.1 follows by setting $\epsilon_0 = \lambda_0^{\frac{N+2}{N-2}}$ and $\epsilon = \lambda^{\frac{N+2}{N-2}}$.

4.3 An existence result for σ and π small in L^2

In this section, we adapt the method of [Holst *et al.*, 2008, Holst *et al.*, 2009, Maxwell, 2009] to our context. The first step is to prove an existence result for solutions to the Lichnerowicz equation (3a) in the presence of a scalar field. Very nice existence results for solutions to this equation are given in [Hebey *et al.*, 2008], [Premoselli, 2014, Premoselli, 2015] and [Hebey et Veronelli, 2014]. We prove here an existence result suited to our applications. See Theorem 4.3.2. We then study the full system (3) and obtain the following theorem:

Theorem 4.3.1. *Let data be given on M as specified in (4), and assume that conditions (5) hold. Assume further that the operator $L_{g,\psi}$ is coercive. Then the system (3) admits at least one solution*

$(\phi, W) \in W^{2,\frac{p}{2}} \times W^{2,\frac{p}{2}}$ *provided that*

$$\int \left(|\sigma|^2 + \pi^2\right) d\mu^g$$

is less than some small constant (depending on the remaining seed data).

4.3.1 The Lichnerowicz equation

Here and in what follows, we define the following norm. Given $\phi \in H^1(M, g)$, we set

$$\|\phi\|_h^2 := \int_M \left(\frac{4(n-1)}{n-2} |d\phi|^2 + \mathcal{R}_\psi \phi^2\right) d\mu^g.$$

Since we assumed that the modified conformal Laplacian is coercive, there exists a constant $s > 0$ such that for any $\phi \in H^1$, we have

$$\|\phi\|_h^2 \geqslant s\, \|\phi\|_{L^N}^2 . \tag{4.6}$$

The aim of this section is to prove the following theorem:

Theorem 4.3.2. *Assuming that* $|\sigma + \mathbb{L}W|^2 + \pi^2 \in L^{\frac{p}{2}}$, *there exists a (small) constant* $\mu = \mu(s, \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}) >$ 0 *such that if*

$$0 < \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g < \mu$$

the Lichnerowicz equation (3a) *admits a solution* $\phi \in H^1$ *which is a stable minimizer for the functional*

$$\begin{aligned} I_W(\phi) := \frac{1}{2} \int_M \left(\frac{4(n-1)}{n-2} |d\phi|^2 + \mathcal{R}_\psi \phi^2\right) d\mu^g - \int_M \frac{\mathcal{B}_{\tau,\psi}}{N} \phi^N d\mu^g \\ + \int_M \frac{|\sigma + \mathbb{L}W|^2 + \pi^2}{N \phi^N} d\mu^g. \end{aligned} \tag{4.7}$$

and whose energy satisfies

$$\|\phi\|_h^2 \leqslant C \left(\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g\right)^{\frac{2}{N+2}} .$$

for some constant $C = C(s, \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}, \mu)$.

A much more detailed analysis of the Lichnerowicz equation was performed in [Premoselli, 2015] assuming that the coefficients are continuous and $3 \leqslant n \leqslant 5$. The spirit of the proof of Theorem 4.3.2 is different from [Hebey *et al.*, 2008]. The point being that we want to obtain a stable solution ϕ_0, meaning that ϕ_0 is a stable local minimum for the functional I defined in (4.7), while [Hebey *et al.*, 2008] uses the mountain pass lemma. Stability will ensure that the minimum ϕ_0 varies continously with respect to the parameters. This will turn out to be very important when applying the Schauder fixed point theorem in Section 4.3.2.

The proof of Theorem 4.3.2 will be carried out in the remainder of this section. For convenience, we denote

$$A_W := |\sigma + \mathbb{L}W|^2 + \pi^2.$$

We also denote by B_{R_0} the ball of radius $R_0 > 0$ centered at the origin in H^1 for the norm $\|\cdot\|_h$.

Lemma 4.3.3. *There exists an* $R_0 > 0$ *depending only on* g *and* ψ *such that the functional*

$$\overline{I}(\phi) := \frac{1}{2}\int_M \left(\frac{4(n-1)}{n-2}|d\phi|^2 + \mathcal{R}_\psi \phi^2\right) d\mu^g - \int_M \frac{\mathcal{B}_{\tau,\psi}}{N}|\phi|^N d\mu^g \tag{4.8}$$

has $\mathrm{Hess}\,\overline{I}(\phi)(u,u) \geqslant \frac{1}{2}\|u\|_h^2$ *for all* $\phi \in B_{R_0}(0)$ *and all* $u \in H^1$.

In particular, we have

$$\frac{1}{4}\|\phi\|_h^2 \leqslant \overline{I}(\phi) \tag{4.9}$$

for all $\phi \in B_{R_0}$. Indeed, applying the Taylor-Lagrange theorem to the function

$$f(t) := \overline{I}(t\phi),$$

we get that there exists $t_0 \in (0,1)$ such that

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(t_0).$$

Since $f(0) = f'(0) = 0$ and $f''(t_0) = \mathrm{Hess}\,\overline{I}(t_0\phi)(\phi,\phi) \geqslant \frac{1}{2}\|\phi\|_h^2$, we obtain Estimate (4.9).

Proof of Lemma 4.3.3. The Hessian of $\overline{I}$ at $\phi \in H^1$ and in the direction $u \in H^1$ is given by

$$\mathrm{Hess}\,\overline{I}(\phi)(u,u) = \int_M \left[\frac{4(n-1)}{n-2}|du|^2 + \mathcal{R}_\psi u^2 - (N-1)\mathcal{B}_{\tau,\psi}|\phi|^{N-2}u^2\right] d\mu^g.$$

We estimate the Hessian as follows:

$$\begin{aligned}
\mathrm{Hess}\,\overline{I}(\phi)(u,u) &\geqslant \int_M \left(\frac{4(n-1)}{n-2}|du|^2 + \mathcal{R}_\psi u^2\right) d\mu^g - (N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\|\phi\|_{L^N}^{N-2}\|u\|_{L^N}^2 \\
&\geqslant \|u\|_h^2 - \frac{N-1}{s}\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\|\phi\|_{L^N}^{N-2}\|u\|_h^2 \\
&\geqslant \left(1 - \frac{N-1}{s}\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\|\phi\|_{L^N}^{N-2}\right)\|u\|_h^2.
\end{aligned}$$

Thus, if $\|\phi\|_{L^N} < \left(\frac{s}{2(N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}}\right)^{\frac{1}{N-2}}$, the Hessian of $\overline{I}$ satisfies the assumptions of the lemma. From Equation (4.6), the conclusion of the lemma holds with

$$R_0 = s^{1/2}\left(\frac{s}{2(N-1)\mathcal{B}_{\tau,\psi}}\right)^{\frac{1}{N-2}}. \tag{4.10}$$

□

We now introduce the following functional:

$$\begin{aligned}
I_W^\epsilon(\phi) :=& \frac{1}{2}\int_M \left(|d\phi|^2 + \mathcal{R}_\psi \phi^2\right) d\mu^g - \int_M \frac{\mathcal{B}_{\tau,\psi}}{N}|\phi|^N d\mu^g \\
&+ \int_M \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g + \int_M \phi_-^N d\mu^g,
\end{aligned} \tag{4.11}$$

where $\phi_- := -\min\{\phi, 0\}$. Note that the two terms we added are convex on the set

$$\Omega_\epsilon := \{\phi \in H^1, \phi \geqslant -\epsilon/2 \text{ a.e.}\}. \tag{4.12}$$

This set is convex and closed for the H^1-norm. Indeed, we have

$$\Omega_\epsilon = \bigcap_f \left\{\phi \in H^1, \int_M f\phi d\mu^g \geqslant -\frac{\epsilon}{2}\int_M f d\mu^g\right\},$$

where we took the intersection over the set of (say) continuous positive functions f. In particular, the set Ω_ϵ is compact for the weak topology on H^1.

Continuity of I_W^ϵ is easy to prove. Indeed, the only difficult term to prove continuity of is

$$\int_M \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g$$

But, given $\phi_0 \in \Omega_\epsilon$ and $\nu > 0$, there exists $\Lambda > 0$ so that

$$\frac{1}{N}\left(\frac{2}{\epsilon}\right)^N \int_{A_W \geqslant \Lambda} A_W d\mu^g \leqslant \frac{\nu}{4}.$$

So, for any $\phi \in \Omega_\epsilon$, we have

$$\begin{aligned}
&\left|\int_M \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g - \int_M \frac{A_W}{N(\phi_0+\epsilon)^N} d\mu^g\right| \\
&\qquad \leqslant \left|\int_{A_W<\Lambda} \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g - \int_{A_W<\Lambda} \frac{A_W}{N(\phi_0+\epsilon)^N} d\mu^g\right| \\
&\qquad\quad + \left|\int_{A_W\geqslant\Lambda} \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g\right| + \left|\int_{A_W\geqslant\Lambda} \frac{A_W}{N(\phi_0+\epsilon)^N} d\mu^g\right| \\
&\qquad \leqslant \left|\int_{A_W<\Lambda} \left(\frac{A_W}{N(\phi+\epsilon)^N} - \frac{A_W}{N(\phi_0+\epsilon)^N}\right) d\mu^g\right| + \frac{\nu}{2} \\
&\qquad \leqslant \left|\int_{A_W<\Lambda} \left(\int_0^1 \frac{1}{(t\phi + (1-t)\phi_0 + \epsilon)^{N+1}} dt\right)(\phi - \phi_0) A_W d\mu^g\right| + \frac{\nu}{2} \\
&\qquad \leqslant \Lambda \epsilon^{-N-1} \|\phi - \phi_0\|_{L^1} + \frac{\nu}{2}.
\end{aligned}$$

Hence, provided $\|\phi - \phi_0\|_{L^1} < \frac{\nu}{2\Lambda}\epsilon^{N+1}$, we have

$$\left|\int_M \frac{A_W}{N(\phi+\epsilon)^N} d\mu^g - \int_M \frac{A_W}{N(\phi_0+\epsilon)^N} d\mu^g\right| < \nu.$$

The H^1-norm being stronger than the L^1-norm this concludes the proof of the continuity of I_W^ϵ. Note that I itself is continuous a priori only on $\bigcup_{\epsilon<0} \Omega_\epsilon$ which is not closed. This is one of the reasons why we need to regularize I.

Now note that since I_W^ϵ is (strictly) convex and continuous on $\Omega_\epsilon \cap B_{R_0}$ it is weakly lower semi-continuous. In particular, there exists a unique $\phi_\epsilon \in \Omega_\epsilon \cap B_{R_0}$ such that

$$I_W^\epsilon(\phi_\epsilon) = \inf_{\phi \in \Omega_\epsilon \cap B_{R_0}} I_W^\epsilon(\phi).$$

The ϕ_--term in the definition of I_W^ϵ together with the strict convexity of the functional I_W^ϵ ensures that $\phi_\epsilon \geqslant 0$. Indeed, we see that $I_W^\epsilon(\phi) \geqslant I_W^\epsilon(|\phi|)$ where the inequality is strict unless $\phi \geqslant 0$ a.e.. It follows from elliptic regularity that $\phi_\epsilon \in W^{2,\frac{p}{2}}$ and from the Harnack inequality that $\phi_\epsilon > 0$. In particular $\phi_\epsilon \in \Omega_0 \cap B_{R_0}$.

To estimate the norm of ϕ_ϵ, we evaluate I_W^ϵ on constant functions $\phi \equiv \lambda > 0$:

$$\begin{aligned}
I_W^\epsilon(\lambda) &= \frac{\lambda^2}{2}\int_M \mathcal{R}_\psi d\mu^g - \frac{\lambda^N}{N}\int_M \mathcal{B}_{\tau,\psi} d\mu^g + \frac{(\lambda+\epsilon)^{-N}}{N}\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g \\
&= \frac{a}{2}\lambda^2 - \frac{b}{N}\lambda^N + \frac{c}{N}(\lambda+\epsilon)^{-N},
\end{aligned}$$

where

$$a = \int_M \mathcal{R}_\psi d\mu^g, \qquad b = \int_M \mathcal{B}_{\tau,\psi} d\mu^g, \qquad c = \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g.$$

Some simple analysis shows that the stable minimum of $I_0(\lambda)$ is attained at some value $\lambda \sim \left(\frac{c}{a}\right)^{\frac{1}{N+2}}$. We thus have

$$\begin{aligned}
\bar{I}(\phi_\epsilon) &\leqslant I_W^\epsilon(\phi_\epsilon) \\
&\leqslant I_W^\epsilon\left(\left(\frac{c}{a}\right)^{\frac{1}{N+2}} - \epsilon\right) \\
&\leqslant \left(\frac{1}{2} + \frac{1}{N}\right)\left(\frac{c^2}{a^N}\right)^{\frac{1}{N+2}} - \frac{b}{N}\left(\frac{c}{a}\right)^{\frac{N}{N+2}} - a\epsilon\left(\frac{c}{a}\right)^{\frac{1}{N+2}} + \frac{a}{2}\epsilon^2.
\end{aligned}$$

Choosing $\epsilon \leqslant \left(\frac{c}{a}\right)^{\frac{1}{N+2}}$ and using Inequality (4.9), we get

$$\frac{1}{4}\|\phi_\epsilon\|_h^2 \leqslant \bar{I}(\phi_\epsilon) \leqslant \left(\frac{1}{2} + \frac{1}{N}\right)\left(\frac{c^2}{a^N}\right)^{\frac{1}{N+2}}.$$

It is important to remark at this point that the estimate we got for $\|\phi_\epsilon\|_h^2$ is actually independent of ϵ.

Following [Premoselli, 2014], we construct a (positive) sub-solution to the equation for the critical points of the functional (4.11):

$$\frac{4(n-1)}{n-2}\Delta\phi + \mathcal{R}_\psi\phi = \mathcal{B}_{\tau,\psi}\phi^{N-1} + \frac{|\sigma + \mathbb{L}W|^2 + \pi^2}{(\phi+\epsilon)^{N+1}}. \tag{4.13}$$

Note that the set Ω_ϵ has empty interior in H^1 so one cannot speak about the Hessian of I_W^ϵ restricted to this set. Critical points are here to be understood as points for which the Gâteaux derivative of the functional I_W^ϵ vanishes in the direction of smooth functions. Nevertheless Equation (4.13) is satisfied by the function ϕ_ϵ as long as $\|\phi_\epsilon\|_h < R_0$ because one then has that $\phi_\epsilon + t\xi \in \Omega_\epsilon \cap B_{R_0}$ for any smooth function ξ as long as $|t|$ is small enough.

Since the construction of a subsolution will be useful later, we collect it in a lemma:

Lemma 4.3.4. *There exists a positive subsolution $\phi_{\mathrm{sub}} \in W^{2,\frac{p}{2}}$ independent of ϵ to all (4.13). Further, ϕ_{sub} can be chosen as small as we want in H^1. If ϕ_{sub} and ϕ_ϵ satisfy*

$$\|\phi_\epsilon\|_{L^N}, \|\phi_{\mathrm{sub}}\|_{L^N} < \left(\frac{s}{(N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}}\right)^{\frac{N}{N-2}},$$

we have $\phi_\epsilon \geqslant \phi_{\mathrm{sub}}$.

Proof. Defining $\mathcal{B}_- := \min\{\mathcal{B}_{\tau,\psi}, 0\}$, and given some α to be chosen later, we solve the following equation for u:

$$\frac{4(n-1)}{n-2}\Delta u + \mathcal{R}_\psi u = |\sigma + \mathbb{L}W|^2 + \pi^2 + \alpha\mathcal{B}_-. \tag{4.14}$$

Note that if $\alpha = 0$, this equation was already studied in Step 1, Section 4.2. The corresponding solution u was continuous and positive; hence, choosing $\alpha > 0$ small enough, we still get a positive solution to (4.14).

We now set $\phi_{\mathrm{sub}} := \theta u$ for some $\theta > 0$. As in the proof of Theorem 4.2.1, it can be checked that, provided θ is small enough (depending only on $\max(u)$), ϕ_{sub} is a subsolution to (4.13), namely:

$$\frac{4(n-1)}{n-2}\Delta\phi_{\mathrm{sub}} + \mathcal{R}_\psi\phi_{\mathrm{sub}} \leqslant \mathcal{B}_{\tau,\psi}\phi_{\mathrm{sub}}^{N-1} + \frac{|\sigma + \mathbb{L}W|^2 + \pi^2}{(\phi_{\mathrm{sub}} + \epsilon)^{N+1}}.$$

Indeed, the condition for ψ_{sub} to be a subsolution reads

$$\left(\theta - \theta^{-N-1}u^{-N-1}\right)A_W + \alpha\theta\mathcal{B}_- - \mathcal{B}_{\tau,\psi}\theta^{N-1}u^{N-1} \leqslant 0,$$

which follows from

$$\theta\left(1 - \theta^{-N-2}u^{-N-1}\right)A_W + \mathcal{B}_-\left(\alpha\theta - \theta^{N-1}u^{N-1}\right) \leqslant 0.$$

This last condition is fulfilled by choosing $\theta > 0$ such that

$$\left\{\begin{array}{ccl} 1 & \leqslant & \theta^{-N-2}u^{-N-1}, \\ \alpha\theta & \geqslant & \theta^{N-1}u^{N-1} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \theta \leqslant (\max u)^{\frac{N+1}{N+2}}, \\ \theta \leqslant \alpha^{\frac{1}{N-2}}(\max u)^{\frac{1-N}{N-2}}. \end{array}\right.$$

We define $(\phi_\epsilon - \phi_{\mathrm{sub}})_- := \min\{0, \phi_\epsilon - \phi_{\mathrm{sub}}\}$. Subtracting Equation (4.13) for ϕ_ϵ with the previous inequality satisfied by ϕ_{sub}, multiplying by $(\phi_\epsilon - \phi_{\mathrm{sub}})_-$ and integrating over M, we get:

$$\begin{aligned}
&\int_M \left(\frac{4(n-1)}{n-2}|d(\phi_\epsilon - \phi_{\mathrm{sub}})_-|^2 + \mathcal{R}_\psi(\phi_\epsilon - \phi_{\mathrm{sub}})_-^2\right) d\mu^g \\
&\qquad \leqslant \int_M \mathcal{B}_{\tau,\psi}\left(\phi_\epsilon^{N-1} - \phi_{\mathrm{sub}}^{N-1}\right)(\phi_\epsilon - \phi_{\mathrm{sub}})_- d\mu^g \\
&\qquad\qquad + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)\left(\frac{1}{(\phi_\epsilon + \epsilon)^{N+1}} - \frac{1}{(\phi_{\mathrm{sub}} + \epsilon)^{N+1}}\right)(\phi_\epsilon - \phi_{\mathrm{sub}})_- d\mu^g, \\
&\qquad \leqslant (N-1)\int_M \mathcal{B}_{\tau,\psi}\left(\int_0^1 (t\phi_\epsilon + (1-t)\phi_{\mathrm{sub}})^{N-2} dt\right)(\phi_\epsilon - \phi_{\mathrm{sub}})_-^2 d\mu^g \\
&\qquad\qquad - (N+1)\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)\left(\int_0^1 (t\phi_\epsilon + (1-t)\phi_{\mathrm{sub}} + \epsilon)^{-N-2} dt\right)(\phi_\epsilon - \phi_{\mathrm{sub}})_-^2 d\mu^g, \\
&\qquad \leqslant (N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\left(\int_M \left(\int_0^1 (t\phi_\epsilon + (1-t)\phi_{\mathrm{sub}})^{N-2} dt\right)^{\frac{N}{N-2}} d\mu^g\right)^{\frac{N-2}{N}} \\
&\qquad\qquad\qquad \times \left(\int_M (\phi_\epsilon - \phi_{\mathrm{sub}})_-^N d\mu^g\right)^{\frac{2}{N}},
\end{aligned}$$

$$\begin{aligned}
&s\left(\int_M (\phi_\epsilon - \phi_{\mathrm{sub}})_-^N d\mu^g\right)^{\frac{2}{N}} \\
&\qquad \leqslant (N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\left(\max\{\|\phi_\epsilon\|_{L^N}, \|\phi_{\mathrm{sub}}\|_{L^N}\}\right)^{\frac{N-2}{N}}\left(\int_M (\phi_\epsilon - \phi_{\mathrm{sub}})_-^N d\mu^g\right)^{\frac{2}{N}}.
\end{aligned}$$

We conclude that if $(N-1)\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}\left(\max\{\|\phi_\epsilon\|_{L^N}, \|\phi_{\mathrm{sub}}\|_{L^N}\}\right)^{\frac{N-2}{N}} < s$, $(\phi_\epsilon - \phi_{\mathrm{sub}})_- \equiv 0$ which is equivalent to saying that $\phi_\epsilon \geqslant \phi_{\mathrm{sub}}$. ◻

We now let ϵ go to zero. From the fact that $\Omega_0 \cap B_{R_0}$ is weakly compact, there exists $\phi_0 \in \Omega_0 \cap B_{R_0}$ which is the weak limit of some sequence $(\phi_{\epsilon_i})_{i\geqslant 0}$, where $\epsilon_i \to 0$. We can also assume that $\phi_{\epsilon_i} \to \phi_0$ a.e..

Since all ϕ_ϵ are greater than or equal to ϕ_{sub}, we have $\phi_0 \geqslant \phi_{\mathrm{sub}}$ and

$$\frac{|\sigma + \mathbb{L}W|^2 + \pi^2}{(\phi_{\epsilon_i} + \epsilon_i)^{N+1}} \to \frac{|\sigma + \mathbb{L}W|^2 + \pi^2}{\phi_0^{N+1}} \text{ in } L^q$$

for any $q < \frac{p}{2}$ since

$$\frac{1}{(\phi_{\epsilon_i} + \epsilon_i)^{N+1}}$$

is uniformly bounded in L^∞ and

$$\frac{1}{(\phi_{\epsilon_i} + \epsilon_i)^{N+1}} \to \frac{1}{\phi_0^{N+1}} \text{ a.e..}$$

As a consequence ϕ_0 satisfies the Lichnerowicz equation (3a) in a weak sense. Elliptic regularity shows that $\phi_0 \in W^{2,\frac{p}{2}}$ and $I_W^\epsilon(\phi_\epsilon) \to I(\phi_0)$. Since $I_W^\epsilon \leqslant I$ on $B_{R_0} \cap \Omega_0$, we have

$$I_W^\epsilon(\phi_\epsilon) = \min_{B_{R_0}\cap\Omega_0} I_W^\epsilon \leqslant \inf_{B_{R_0}\cap\Omega_0} I.$$

This means that $I(\phi_0) = \inf_{B_{R_0}\cap\Omega_0} I$: ϕ_0 is a minimizer for I. From the fact that $B_{R_0} \cap \Omega_0$ is convex and I is strictly convex on $B_{R_0} \cap \Omega_0$, we deduce that ϕ_0 is the unique positive solution to the Lichnerowicz equation on B_{R_0}.

4.3.2 The coupled system

We now study the coupled system. As in [Dahl *et al.*, 2012], the operator

$$\phi \mapsto \frac{3n-2}{n-1}\Delta\phi + \mathcal{R}_\psi\phi$$

naturally appears. We make the assumption that this operator is coercive and let s' be some positive constant so that

$$\|u\|_k^2 := \int_M \left(\frac{3n-2}{n-1}|du|^2 + \mathcal{R}_\psi u^2\right) d\mu^g \geqslant s' \|u\|_{L^N}^2 . \tag{4.15}$$

We shall even assume that $\mathcal{R}_\psi$ is positive. This assumption can be removed by performing a conformal change of the metric g; see [Choquet-Bruhat *et al.*, 2007b, Proposition 1], and working with the conformal thin sandwich method which is explicitly conformally covariant and differs from the conformal method by the introduction of a lapse function. We refer the reader to [Maxwell, 2014] for an extensive discussion of this fact.

It should be noted however that the proof we present here uses the York splitting. Namely for any TT-tensor σ and any 1-form W, we have

$$\int_M \langle\sigma, \mathbb{L}W\rangle \, d\mu^g = 0;$$

i.e., the set of TT-tensors is L^2-orthogonal to the image of $\mathbb{L}$. This has the following consequence

$$\int_M |\sigma + \mathbb{L}W|^2 \, d\mu^g = \int_M |\sigma|^2 \, d\mu^g + \int_M |\mathbb{L}W| \, d\mu^g$$

that is used in establishing Estimate (4.16).

In the conformal thin sandwich method, $\mathbb{L}W$ is replaced by $\frac{1}{2\eta}\mathbb{L}W$ both in the Lichnerowicz equation (3a) and in the vector equation (3b), where η is the lapse function; see [Maxwell, 2014, Section 6]. In particular, the previous identities do not apply. Still, we can rely on the estimate

$$\int_M |\sigma + \mathbb{L}W|^2 \, d\mu^g \leqslant 2\left(\int_M |\sigma|^2 \, d\mu^g + \int_M |\mathbb{L}W| \, d\mu^g\right).$$

to get an analogue to Estimate (4.16) with a coefficient 2 appearing in front of $\int_M |\sigma|^2 d\mu^g$.

We are going to use a fixed point argument. Starting from $\phi_0 \in L^{Np}$, we solve the vector equation (3b) with $\phi \equiv \phi_0$ and get $W \in W^{2,q}$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$ which we plug in the Lichnerowicz equation. Assuming that $\mathbb{L}W$ is small enough in L^2, Theorem 4.3.2 yields a unique $\phi > 0$ in $B_{R_0} \subset H^1$ which, by elliptic regularity, belongs to $W^{2,\frac{p}{2}} \subset L^\infty \subset L^{Np}$. We call this mapping (wherever it is defined) Φ.

We first prove the following lemma:

Lemma 4.3.5. *There exists a $\mu_0 > 0$ and a constant $R > 0$ such that, provided*

$$\int_M \left(|\sigma|^2 + \pi^2\right) d\mu^g < \mu_0,$$

the mapping Φ is well defined on the set

$$C := \left\{\phi \in L^{Np}, \int_M \phi^{2N} d\mu^g \leqslant R\right\}$$

and C is invariant under the mapping Φ.

In the course of the proof, we will use the following fact: There exists a constant $\gamma > 0$ such that for any $W \in H^1$, we have

$$\int_M |\mathbb{L}W|^2\, d\mu^g \geqslant \gamma \left(\int_M |W|^N\, d\mu^g\right)^{2/N}.$$

Proof. We contract the vector equation with W and integrate over M. We obtain:

$$\begin{aligned}
\frac{1}{2}\int_M |\mathbb{L}W|^2\, d\mu^g &= -\frac{n-1}{n}\int_M \phi^N \langle d\tau, W\rangle + \int_M \pi \langle d\psi, W\rangle\, d\mu^g\\
&\leqslant \frac{n-1}{2n}\left(\alpha \int_M \phi_0^{2N} d\mu^g + \frac{1}{\alpha}\int_M |d\tau|^2\, |W|^2\, d\mu^g\right)\\
&\quad + \frac{1}{2}\left(\beta \int_M \pi^2 d\mu^g + \frac{1}{\beta}\int_M |d\psi|^2\, |W|^2\, d\mu^g\right)\\
&\leqslant \frac{n-1}{2n}\left[\alpha \int_M \phi_0^{2N} d\mu^g + \frac{\|d\tau\|_{L^n}^{2/n}}{\alpha}\left(\int_M |W|^N\, d\mu^g\right)^{2/N}\right]\\
&\quad + \frac{1}{2}\left[\beta \int_M \pi^2 d\mu^g + \frac{\|d\psi\|_{L^n}^{2/n}}{\beta}\left(\int_M |W|^N\, d\mu^g\right)^{2/N}\right]\\
&\leqslant \frac{n-1}{2n}\left[\alpha \int_M \phi_0^{2N} d\mu^g + \frac{\|d\tau\|_{L^n}^{2/n}}{\alpha\gamma}\int_M |\mathbb{L}W|^2\, d\mu^g\right]\\
&\quad + \frac{1}{2}\left[\beta \int_M \pi^2 d\mu^g + \frac{\|d\psi\|_{L^n}^{2/n}}{\beta\gamma}\int_M |\mathbb{L}W|^2\, d\mu^g\right],
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{n-1}{2n}\frac{\|d\tau\|_{L^n}^{2/n}}{\alpha\gamma} - \frac{1}{2}\frac{\|d\psi\|_{L^n}^{2/n}}{\beta\gamma}\right)\int_M |\mathbb{L}W|^2\, d\mu^g\\
&\qquad\qquad \leqslant \frac{n-1}{2n}\alpha \int_M \phi_0^{2N} d\mu^g + \frac{1}{2}\beta \int_M \pi^2 d\mu^g.
\end{aligned}$$

Choosing α and β large enough, we conclude that there exist constants c_1 and c_2 such that

$$\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g \leqslant \int_M |\sigma|^2 d\mu^g + c_1 \int_M \phi_0^{2N} d\mu^g + (1 + c_2)\int_M \pi^2 d\mu^g. \tag{4.16}$$

This proves that, if $\phi_0 \in L^{2N}$ and $\pi, \sigma \in L^2$ are small enough, $\mathbb{L}W$ is small in L^2 so Theorem 4.3.2 applies; giving a solution ϕ to the Lichnerowicz equation.

Next, we multiply the Lichnerowicz equation by ϕ^{N+1} and integrate by parts the Laplacian:

$$\begin{aligned}
&\frac{3n-2}{n-1}\int_M \left|d\phi^{\frac{N}{2}+1}\right|^2 d\mu^g + \int_M \mathcal{R}_\psi \phi^{N+2} d\mu^g \\
&\qquad = \int_M \mathcal{B}_{\tau,\psi}\phi^{2N} d\mu^g + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g \\
&\qquad \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} \int_M \phi^{2N} d\mu^g + \int_M \left(|\sigma|^2 + (1+c_2)\pi^2\right) d\mu^g + c_1 \int_M \phi_0^{2N} d\mu^g \\
&\qquad \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} \left(\int_M \phi^N d\mu^g\right)^{\frac{N-2}{N}} \left(\int_M \phi^{N\frac{N+2}{2}} d\mu^g\right)^{\frac{2}{N}} \\
&\qquad\qquad + \int_M \left(|\sigma|^2 + (1+c_2)\pi^2\right) d\mu^g + c_1 \int_M \phi_0^{2N} d\mu^g.
\end{aligned}$$

Hence, introducing the norm $\|\cdot\|_k$ (see (4.15)),

$$\begin{aligned}
&\left(1 - \frac{\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}}{(s')^{1/N}} \left(\int_M \phi^N d\mu^g\right)^{\frac{N-2}{N}}\right) \left\|\phi^{\frac{N}{2}+1}\right\|_k^2 \\
&\qquad \leqslant \int_M \left(|\sigma|^2 + (1+c_2)\pi^2\right) d\mu^g + c_1 \int_M \phi_0^{2N} d\mu^g.
\end{aligned} \tag{4.17}$$

From the Sobolev embedding together with the Hölder inequality, we get:

$$\begin{aligned}
\int_M \phi^{2N} d\mu^g &\leqslant \mathrm{Vol}(M,g)^{\frac{N-2}{N+2}} \left\|\phi^{1+\frac{N}{2}}\right\|_{L^N}^{\frac{4N}{N+2}} \\
&\leqslant \mathrm{Vol}(M,g)^{\frac{N-2}{N+2}} (s')^{-\frac{2N}{N+2}} \left\|\phi^{1+\frac{N}{2}}\right\|_k^{\frac{4N}{N+2}}.
\end{aligned}$$

Thus, assuming that $\|\phi\|_{L^N}$ is small enough

$$\left(\int_M \phi^N d\mu^g\right)^{\frac{N-2}{N}} \leqslant \frac{1}{2} \frac{(s')^{1/N}}{\left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty}},$$

we conclude that

$$\begin{aligned}
&\frac{s'}{2} \mathrm{Vol}(M,g)^{\frac{N-2}{2N}} \left(\int_M \phi^{2N} d\mu^g\right)^{\frac{N+2}{2N}} \\
&\qquad \leqslant \int_M \left(|\sigma|^2 + (1+c_2)\pi^2\right) d\mu^g + c_1 \int_M \phi_0^{2N} d\mu^g.
\end{aligned}$$

Denoting

$$y = \int_M \phi^{2N} d\mu^g, \text{ resp. } y_0 = \int_M \phi_0^{2N} d\mu^g,$$

we obtain an inequality of the following form for y

$$y \leqslant \left(\frac{x}{\lambda} + \frac{c_1}{\lambda} y_0\right)^{\frac{2N}{N+2}}, \tag{4.18}$$

where
$$x = \int_M \left(|\sigma|^2 + (1+c_2)\pi^2\right) d\mu^g \text{ and } \lambda = \frac{s'}{2} \operatorname{Vol}(M,g)^{\frac{N-2}{2N}}.$$
We denote by $f(y_0)$ the right-hand side of (4.18). Note that f is an increasing function. We seek R such that $R > 0$, $R \ll 1$ and $f(R) \leqslant R$. This would have the consequence that the set
$$C = \left\{\phi \in L^{Np}, \int_M \phi^{2N} d\mu^g \leqslant R\right\}$$
is stable for the mapping Φ. Indeed, we would then have that, given $\phi_0 \in C$,
$$\int_M \phi^{2N} d\mu^g \leqslant f\left(\int_M \phi_0^{2N} d\mu^g\right) \leqslant f(R) \leqslant R.$$
From a simple Taylor expansion, we see that $R = 2\left(\frac{x}{\lambda}\right)^{\frac{2N}{N+2}}$ fulfills the inequality $f(R) \leqslant R$ provided that x is small enough. □

The remaining steps of the proof go as in [Maxwell, 2009]. There is however a subtlety appearing here. Continuity of the solution ϕ of the Lichnerowicz equation (3a) with respect to $\mathbb{L}W$ is usually obtain by the implicit function theorem. But the set $\Omega_0 = \{\phi \in H^1, \phi \geqslant 0 \text{ a.e.}\}$ has empty interior. Hence working on the set C is not enough.

Proposition 4.3.6. *Assuming that*
$$\int_M \left(|\sigma|^2 + \pi^2\right) d\mu^g < \mu_0,$$
where $\mu_0 > 0$ is as defined in Lemma 4.3.5, there exist sequences $(q_i)_{i\geqslant 0}$ and $(R_i)_{i\geqslant 0}$, $q_i \geqslant 2$, $q_i \to \infty$ and $R_i > 0$ such that, setting
$$C_k := C \cap \bigcap_{i=0}^{k} \left\{\phi \in L^{Nq_i}, \|\phi\|_{L^{Nq_i}} \leqslant R_i \text{ for all } i, 0 \leqslant i \leqslant k\right\},$$
Φ maps C_k into $C_{k+1} \subset C_k$.

We use an induction argument which is quite similar in spirit to the one used in [Dahl *et al.*, 2012, Gicquaud et Sakovich, 2012]. Note that, however, in these references, the Laplacian term is discarded because it vanishes for large solutions. Here it will play an important role.

Proof of Proposition 4.3.6. We define sequences $q_i \geqslant 2, R_i$ inductively so that
$$\sup\left\{\|\phi\|_{L^{Nq_i}}, \phi \in \Phi^i(C)\right\} \leqslant R_i.$$
From Lemma 4.3.5, we can choose $q_0 = 2$ and $R_0 = R^{1/2N}$.

Given $\phi_0 \in C_i$, we set $\phi = \Phi(\phi_0)$. Note that $\phi_0 \in C_{i-1}$ (or $\phi_0 \in C$ if $i = 0$); hence, by induction, $\phi \in C_i$ (when $i = 0$, this is Lemma 4.3.5).

The solution W to the vector equation
$$-\frac{1}{2}\mathbb{L}^*\mathbb{L}W = \frac{n-1}{n}\phi^N d\tau + \pi d\psi$$

belongs to W^{2,r_i}, where r_i is given by

$$\frac{1}{r_i} = \frac{1}{n} + \frac{1}{q_i} \geqslant \frac{1}{n}.$$

By elliptic regularity, together with the Sobolev embedding,

$$\begin{aligned} \|\mathbb{L}W\|_{L^{q_i}} &\lesssim R_i \|d\tau\|_{L^n} + \|\pi d\psi\|_{L^{r_i}} \\ &\lesssim R_i + \|\pi d\psi\|_{L^n}. \end{aligned} \tag{4.19}$$

We multiply the Lichnerowicz equation for $\phi = \Phi(\phi_0)$ by ϕ^{N+1+2k_i} for some $k_i \geqslant 0$ to be chosen later and integrate over M to get:

$$\begin{aligned} &\frac{4(n-1)}{n-2} \frac{N+1+2k_i}{\left(\frac{N}{2}+1+k_i\right)^2} \int_M \left|d\left(\phi^{\frac{N}{2}+1+k_i}\right)\right|^2 d\mu^g + \int_M \mathcal{R}_\psi \phi^{N+2+2k_i} d\mu^g \\ &\qquad = \int_M \mathcal{B}_{\tau,\psi} \phi^{2N+2k_i} d\mu^g + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) \phi^{2k_i}. \end{aligned} \tag{4.20}$$

Since we assumed that $\mathcal{R}_\psi > 0$, there exists a constant $s_i > 0$ so that

$$\forall \xi \in H^1, \frac{4(n-1)}{n-2} \frac{N+1+2k_i}{\left(\frac{N}{2}+1+k_i\right)^2} \int_M |d\xi|^2 d\mu^g + \int_M \mathcal{R}_\xi \xi^2 d\mu^g \geqslant s_i \left(\int_M \xi^N d\mu^g\right)^{\frac{2}{N}}.$$

Applying this inequality to (4.20) with $\xi \equiv \phi^{\frac{N}{2}+1+k_i}$, we get

$$\begin{aligned} &s_i \left(\int_M \phi^{N\left(\frac{N}{2}+1+k_i\right)} d\mu^g\right)^{2/N} \\ &\qquad \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} \int_M \phi^{2N+2k_i} d\mu^g + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) \phi^{2k_i} \\ &\qquad \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} \left(\int_M \phi^{2N} d\mu^g\right)^{1-x} \left(\int_M \phi^{2N+\frac{2k_i}{x}} d\mu^g\right)^x \\ &\qquad\quad + \left(\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)^{\frac{q_i}{2}} d\mu^g\right)^{\frac{2}{q_i}} \left(\int_M \phi^{2k_i \frac{q_i}{q_i-2}} d\mu^g\right)^{1-\frac{2}{q_i}}, \end{aligned} \tag{4.21}$$

where $x \in (0,1)$ is some constant to be chosen later. From Equation (4.19), we have that

$$\left(\int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)^{\frac{q_i}{2}} d\mu^g\right)^{\frac{2}{q_i}} \lesssim \|\sigma\|_{L^p}^2 + \|\pi\|_{L^p}^2 + \|\mathbb{L}W\|_{L^{q_i}}^2$$

is bounded from above independently of W by some constant C_i. We choose k_i so that

$$2k_i \frac{q_i}{q_i - 2} = N q_i;$$

i.e.,

$$k_i = \frac{N}{2}(q_i - 2) \geqslant 0. \tag{4.22}$$

Note that since $\phi = \Phi(\phi_0) \in C_i$, we have that

$$\left(\int_M \phi^{2k_i \frac{q_i}{q_i-2}} d\mu^g\right)^{1-\frac{2}{q_i}} \leqslant R_i^{q_i-2}.$$

We now come back to the choice of x. We let x be such that

$$2N + \frac{2k_i}{x} = N\left(\frac{N}{2} + 1 + k_i\right);$$

that is to say

$$x = \frac{2k_i}{Nk_i + N\left(\frac{N}{2} - 1\right)} < \frac{2}{N}.$$

We finally arrive at the following inequality:

$$\begin{aligned} s_i \left(\int_M \phi^{N\left(\frac{N}{2}+1+k_i\right)} d\mu^g\right)^{2/N} & \\ \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} R^{1-x} & \left(\int_M \phi^{N\left(\frac{N}{2}+1+k_i\right)} d\mu^g\right)^x + C_i R_i^{q_i-2}. \end{aligned}$$

Since $x < \frac{2}{N}$ we immediately deduce that, setting $q_{i+1} = \frac{N}{2} + 1 + k_i$,

$$\|\phi\|_{L^{Nq_{i+1}}} \leqslant R_{i+1}$$

for some R_{i+1} independent of $\phi_0 \in C$, we have

$$q_{i+1} = \frac{N}{2} + 1 + \frac{N}{2}(q_i - 2) = \frac{N}{2}(q_i - 1) + 1$$

so $q_i = 1 + \left(\frac{N}{2}\right)^i$ goes to infinity with i.

We point here that we were slightly sloppy. Namely for $i = 0$, $k_0 = 0$ and $x = 0$ which is not allowed in our calculation. Note however that multiplying the Lichnerowicz equation with ϕ^{N+1} and integrating over M, we get, as in the proof of Lemma 4.3.5, that

$$\begin{aligned} \frac{3n-2}{n-1} \int_M \left|d\phi^{\frac{N}{2}+1}\right|^2 d\mu^g + \int_M \mathcal{R}_\psi \phi^{N+2} d\mu^g & \\ \leqslant \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} \int_M \phi^{2N} d\mu^g + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g & \\ \leqslant R \left\|\mathcal{B}_{\tau,\psi}\right\|_{L^\infty} + \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right) d\mu^g & \end{aligned}$$

so the argument still applies. □

We now choose k so that $q_k \geqslant p$ and set $\overline{C} := C_k$. We come back to the subsolution introduced in Lemma 4.3.4. This lemma is taken from [Maxwell, 2009].

Lemma 4.3.7. *There exists $\eta > 0$ so that all $\phi = \Phi(\phi_0)$ with $\phi_0 \in \overline{C}$ satisfy $\phi \geqslant \eta$.*

Proof. We study in more detail the proof of Lemma 4.3.4. We can write $u = u_1 - \alpha u_2$, where u_1 and u_2 solve

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta u_1 + \mathcal{R}_\psi u_1 = |\sigma + \mathbb{L}W|^2 + \pi^2, \\ \frac{4(n-1)}{n-2}\Delta u_2 + \mathcal{R}_\psi u_2 = \mathcal{B}_+. \end{cases}$$

The Green's function $G(x, y)$ of the modified conformal Laplacian

$$\frac{4(n-1)}{n-2}\Delta + \mathcal{R}_\psi$$

is positive and continuous outside the diagonal of $M \times M$ where it blows up. Hence, there exists a constant $\epsilon > 0$ such that $G(x, y) \geqslant \epsilon$. This implies that

$$\begin{aligned} u_1(x) &= \int_M G(x,y)\left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)(y) d\mu^g(y) \\ &\geqslant \epsilon \int_M \left(|\sigma + \mathbb{L}W|^2 + \pi^2\right)(y) d\mu^g(y) \\ &\geqslant \epsilon \int_M \left(|\sigma|^2 + |\mathbb{L}W|^2 + \pi^2\right) d\mu^g \\ &\geqslant \epsilon \int_M \left(|\sigma|^2 + \pi^2\right) d\mu^g. \end{aligned}$$

So u_1 is bounded from below independently of W so α in the proof of Lemma 4.3.4 can be chosen independently of W so that e.g. $u \geqslant \frac{\epsilon}{2}\int_M \left(|\sigma|^2 + \pi^2\right) d\mu^g$. Since we assumed that $\phi \in \overline{C}$, we also have that

$$|\sigma + \mathbb{L}W|^2 + \pi^2$$

is bounded from above by some constant depending on R' in $L^{p/2}$ so u is bounded in $W^{2,\frac{p}{2}} \hookrightarrow L^\infty$ independently of the choice of $\phi \in \overline{C}$.

Hence, the constant θ so that $\phi_{\mathrm{sub}} = \theta u$ is a sub-solution to the Lichnerowicz equation can be chosen independently of W.

Setting

$$\eta = \frac{\epsilon\theta}{2}\int_M \left(|\sigma|^2 + \pi^2\right) d\mu^g,$$

we have $\phi_{\mathrm{sub}} \geqslant \eta$ so $\phi \geqslant \eta$. □

Lemma 4.3.8. *Under the assumptions of the previous lemma, the mapping $\Phi : \overline{C} \to \overline{C}$ is continuous and compact.*

Proof. We first prove continuity of the mapping Φ. Assume we have a given sequence $(\phi_i)_i$, $\phi_i \in \overline{C}$ such that $\phi_i \to \phi_\infty$ in L^{Np}.

We denote with a prime their images under the mapping Φ: $\phi'_i = \Phi(\phi_i)$, $\phi'_\infty = \Phi(\phi_\infty)$. We also denote by W_i (resp. W_0) the corresponding solutions to the vector equation:

$$\begin{cases} -\frac{1}{2}\mathbb{L}^*\mathbb{L}W_i = \frac{n-1}{n}\phi_i^N d\tau + \pi d\psi, \\ -\frac{1}{2}\mathbb{L}^*\mathbb{L}W_\infty = \frac{n-1}{n}\phi_\infty^N d\tau + \pi d\psi. \end{cases}$$

We have $W_i \to W_\infty$ in $W^{2,q}$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, so $|\mathbb{L}W_i|^2 \to |\mathbb{L}W_\infty|^2$ in $L^{\frac{p}{2}}$. Since the Hessian of I_{W_∞} is more coercive on B_{R_0} than that of $\overline{I}$, we have from Lemma 4.3.3

$$\frac{1}{4}\left\|\phi'_i - \phi'_\infty\right\|_h^2 \leqslant I_{W_\infty}(\phi'_i) - I_{W_\infty}(\phi'_\infty)$$

for some constant $\lambda > 0$. It follows from Lemma 4.3.7 that $\phi'_i \geqslant \eta$ for all i (resp. $\phi'_\infty \geqslant \eta$). As a consequence,

$$\begin{aligned}
\lambda\left\|\phi'_i - \phi'_\infty\right\|_h^2 &\leqslant I_{W_\infty}(\phi'_i) - I_{W_\infty}(\phi'_\infty)\\
&\leqslant \left(I_{W_\infty}(\phi'_i) - I_{W_i}(\phi'_i)\right) + \left(I_{W_i}(\phi'_i) - I_{W_\infty}(\phi'_\infty)\right)\\
&\leqslant \frac{\eta^{-N}}{N}\left\||\mathbb{L}W_i|^2 - |\mathbb{L}W_\infty|^2\right\|_{L^1} + \sup_{\phi\in B_{R_0}\cap\Omega_{-2\eta}}\left|I_{W_i}(\phi) - I_{W_\infty}(\phi)\right|\\
&\leqslant 2\frac{\eta^{-N}}{N}\left\||\mathbb{L}W_i|^2 - |\mathbb{L}W_\infty|^2\right\|_{L^1},
\end{aligned}$$

where to pass from the second line to the third, we used the fact that the map "infimum" is 1-Lipschitzian. Thus we get that $\phi'_i \to \phi'_\infty$ in H^1 and in particular $\phi'_i \to \phi'_\infty$ in L^N. Convergence in L^{Np} follows from elliptic regularity. Indeed, looking at the Lichnerowicz equation for ϕ'_i

$$\frac{4(n-1)}{n-2}\Delta\phi'_i + \mathcal{R}_\psi\phi'_i = \mathcal{B}_{\tau,\psi}(\phi'_i)^{N-1} + \frac{|\sigma + \mathbb{L}W_i|^2 + \pi^2}{(\phi'_i)^{N+1}},$$

we see that the right-hand side is bounded in $L^{\frac{p}{2}}$ independently of i, as a consequence of Lemma 4.3.7. So the sequence (ϕ'_i) is bounded in $W^{2,\frac{p}{2}} \hookrightarrow L^\infty$. By interpolation, (ϕ'_i) is a Cauchy sequence in L^{Np} whose limit in L^N is ϕ'_∞. We conclude that $\phi'_i \to \phi'_\infty$ in L^{Np}.

Compactness of the mapping Φ is fairly simple to verify since we noticed that the set $(\Phi(C))^{\frac{N+2}{2}}$ is bounded in H^1 (this is Estimate (4.17)) so $\Phi(C)$ embeds compactly in L^{2N} by the Rellich theorem. Then notice that pursuing one step further the proof of Proposition 4.3.6, the set $\Phi(\overline{C})$ is bounded in $L^{Nq_{K+1}}$. Compactness of $\Phi(\overline{C})$ for the L^{Np}-norm follows by interpolation. □

Theorem 4.3.1 then follows by applying the Schauder fixed point theorem. Namely, the convex hull of $\Phi(\overline{C}) \subset \overline{C}$ is compact, convex and stable for the mapping Φ. So Φ admits a fixed point $\phi \in \overline{C}$ which is in turn a solution to the conformal constraint equations.

Chapter 5

Existence and Nonuniqueness Results for Solutions to the Einstein-Scalar Field Conformal Constraint Equations

5.1 Introduction

The question of existence of solutions to (3) associated to a given scalar field was studied by many authors. At least in some situations we are successful in extending known results of the vacuum case to the scalar field case. For instance, under some certain assumptions existence is obtained, provided τ is constant, near-constant, or freely chosen as proven in the previous chapter.

Conversely, although nonuniqueness of solutions to (3) in the scalar field case was addressed earlier for the Lichnerowicz equation, that is a special case of (3) when τ is constant (see [Ngo et Xu, 2012], [Premoselli, 2015]). Until now there has been no result providing this property for the (full) scalar field system (3); i.e., τ is non-constant.

This chapter may be understood as an application of both methods introduced in Chapter 2 to the Einstein-scalar field conformal constraint equations. In Section 5.2, we will give another proof of Theorem 4.3.1 using the half-continuity method. In the spirit of Theorem 3.1.1 we will show in the last section that uniqueness of solutions to the scalar-field system (3) fails under some conditions. More precisely, we will prove the following result.

Theorem 5.1.1 (***Nonuniqueness of solutions***)**.** *Let (M, g) be a closed locally conformally flat Riemannian manifold of dimension n, with $3 \leqslant n \leqslant 5$. Assume that the seed data $(V, \tau, \psi, \pi, \sigma)$ given on M are smooth and (M, g) has no conformal Killing vector field. Assume further that $\mathcal{B}_{\tau,\psi} > 0$, and $\frac{4(n-1)}{n-2}\Delta + \mathcal{R}_\psi$ is coercive. If $\pi \not\equiv 0$, then there exists a sequence $\{\epsilon_i\}$ converging to 0 s.t. the system (3) associated to $(g, \tau, \psi, V, \epsilon_i\sigma, \epsilon_i\pi)$ has at least two solutions.*

5.2 Existence of Solutions

In this section we reprove the existence result for solution to the system (3), shown in the previous chapter, by using the half-continuity method introduced in Chapter 2.

Another proof of Theorem 4.3.1. Without loss of generality, we may assume by [Choquet-Bruhat *et al.*, 2007b] and the arguments similar to Remark 1.2.4 that $\mathcal{R}_\psi > 0$.

Let $\kappa^{\frac{1}{N}} = \max\{\|\sigma\|_{L^2}, \|\pi\|_{L^2}\} > 0$ and $a = a(n, g, \psi, \tau, V) > 0$ be a constant to be chosen later. For each $\varphi \in L^\infty_+$, there exists a unique $W_\varphi \in W^{2,p}$ such that

$$-\frac{1}{2}L^*LW_\varphi = \frac{n-1}{n}\varphi^N d\tau - \pi d\psi,$$

and, arguing as in the proof of Theorem 1.2.3 in the case $\mathcal{Y}_g > 0$, there is a unique $\theta \in W^{2,p}_+$ satisfying

$$\frac{4(n-1)}{n-2}\Delta\theta + \mathcal{R}_\psi\theta + \mathcal{B}^-_{\tau,\psi}\theta^{N-1} = \mathcal{B}^+_{\tau,\psi}\varphi^{N-1} + \left(|\sigma + LW|^2 + \pi^2\right)\theta^{-N-1},$$

where

$$\mathcal{B}^+_{\tau,\psi} = \max\left\{\mathcal{B}_{\tau,\psi}, 0\right\} \text{ and } \mathcal{B}^-_{\tau,\psi} = -min\left\{\mathcal{B}_{\tau,\psi}, 0\right\}.$$

Therefore, we may define

$$T(\varphi) = \theta.$$

Similarly to the proof of Proposition 2.2.6, T is proven to be continuous and compact by using the implicit function theorem.

Next, for any given $Q \in \mathbb{N}^*$ we define the map S as follows:

$$S(\varphi) = \begin{cases} \min\{T(\varphi), Q\} & \text{if } \int_M \varphi^{\frac{N(N+2)}{2}} dv \leqslant a\kappa, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Setting

$$\mathscr{C} = \left\{\varphi \in C^0 \ : \ 0 \leqslant \varphi \leqslant Q\right\},$$

it is obvious that since T is compact, $S(\mathscr{C}) \subset \mathscr{C}$ is precompact.

Assume for the moment that S is half-continuous on $\mathscr{C}$. By Corollary 2.3.5, S has a fixed point $\varphi_Q \leqslant Q$. If $\varphi_Q \equiv 0$, then by the definition of S, it follows that $0 = S(0) = \min\{T(0), Q\} > 0$, which is a contradiction. Therefore, from the definition of S we must have that

$$\frac{4(n-1)}{n-2}\Delta\theta_Q + \mathcal{R}_\psi\theta_Q + \mathcal{B}^-_{\tau,\psi}\theta_Q^{N-1} = \mathcal{B}^+_{\tau,\psi}\left(\min\{\theta_Q, Q\}\right)^{N-1} + |\sigma + LW_Q|^2\theta_Q^{-N-1} \tag{5.2a}$$

$$-\frac{1}{2}L^*LW_Q = \frac{n-1}{n}\left(\min\{\theta_Q, Q\}\right)^N d\tau - \pi d\psi, \tag{5.2b}$$

where

$$\varphi_Q = \min\{\theta_Q, Q\} \text{ and } \int_M \varphi_Q^{\frac{N(N+2)}{2}} dv \leqslant a\kappa. \tag{5.3}$$

Now, from the estimate in (5.3), and using an induction argument similar to the one given in the proof of Proposition 4.3.6, we obtain that $\theta_Q \leqslant c_0(n, g, V, \tau, \psi)$, provided κ is small enough (depending only on g, τ, ψ, V and a), and hence the theorem follows if we take $Q > c_0$.

Thus, from arguments above it suffices to show that S is half-continuous on $\mathscr{C}$. In fact, since T is continuous on $\mathscr{C}$, we obtain that so is S at φ s.t. $\int_M \varphi^{\frac{N(N+2)}{2}} dv > a\kappa$ or $\int_M \varphi^{\frac{N(N+2)}{2}} dv < a\kappa$. For the remaining case; i.e., $\int_M \varphi^{\frac{N(N+2)}{2}} dv = a\kappa$, we will show that there exists $m \in M$ s.t. $\varphi(m) > T(\varphi)(m)$. We argue by contradiction. Assume that it is not true; then $T(\varphi) \geqslant \varphi$. In particular,

$$a\kappa = \int_M \varphi^{\frac{N(N+2)}{2}} dv \leqslant \int_M T(\varphi)^{\frac{N(N+2)}{2}} dv. \tag{5.4}$$

We claim first that there exists $c = c(g, V, \tau, \psi) > 0$ such that

$$\|LW_\varphi\|_{L^2} \leqslant c\kappa^{\frac{1}{N}}.$$

Indeed,

$$\begin{aligned}
\|LW_\varphi\|_{L^2} &\leqslant c_1(n, g) \left\| \frac{n-1}{n} \varphi^N d\tau - \pi d\psi \right\|_{L^{\frac{N}{N-1}}} \quad \text{(by Sobolev embedding theorem)} \\
&\leqslant c_2(n, c_1) \left(\left\| \varphi^N d\tau \right\|_{L^{\frac{N}{N-1}}} + \|\pi d\psi\|_{L^{\frac{N}{N-1}}} \right) \\
&\leqslant c_2 \left[\|d\tau\|_{L^p} \left(\int_M \varphi^{\frac{N^2}{N-1}\frac{p}{p-1}} dv \right)^{\frac{N-1}{N}\frac{p-1}{p}} + \|d\psi\|_{L^p} \left(\int_M \pi^{\frac{N}{N-1}\frac{p}{p-1}} dv \right)^{\frac{N-1}{N}\frac{p-1}{p}} \right] \quad \text{(by Hölder inequality)} \\
&\leqslant c_2 \left[\|d\tau\|_{L^p} \left(\int_M \varphi^{\frac{N(N+2)}{2}} dv \right)^{\frac{2}{N+2}} + \|d\psi\|_{L^p} \left(\int_M \pi^2 dv \right)^{\frac{1}{2}} \right] \quad \text{(by Hölder inequality and } p > n) \\
&\leqslant c_3(c_2, \tau, \psi) \left((a\kappa)^{\frac{2}{N+2}} + \kappa^{\frac{1}{N}} \right) \quad \left(\text{by } \int_M \varphi^{\frac{N(N+2)}{2}} dv = a\kappa\right) \\
&\leqslant 2c_3\kappa^{\frac{1}{N}},
\end{aligned} \tag{5.5}$$

where the last inequality holds provided $\kappa \leqslant a^{-n}$.

Now, multiplying the Lichnerowicz equation by $T(\varphi)^{N+1}$ and integrating over M, we obtain

$$\begin{aligned}
\frac{4(n-1)}{n-2} \int_M T(\varphi)^{N+1} \Delta T(\varphi) dv + \int_M \mathcal{R}_\psi T(\varphi)^{N+2} dv + \int_M \mathcal{B}^-_{\tau,\psi} T(\varphi)^{2N} dv &= \int_M \mathcal{B}^+_{\tau,\psi} \varphi^{N-1} T(\varphi)^{N+1} dv \\
&+ \int_M \left(|\sigma + LW|^2 + \pi^2 \right) dv.
\end{aligned} \tag{5.6}$$

Since we assumed that $\mathcal{R}_\psi > 0$, it follows from the Sobolev inequality that

$$\begin{aligned}
\frac{4(n-1)}{n-2} \int_M T(\varphi)^{N+1} \Delta T(\varphi) dv + \int_M \mathcal{R}_\psi T(\varphi)^{N+2} dv &\geqslant \frac{4(N+1)}{N+2} \left\| \nabla T(\varphi)^{\frac{N+2}{2}} \right\|^2_{L^2} + \left(\min \mathcal{R}_\psi \right) \left\| T(\varphi)^{\frac{N+2}{2}} \right\|^2_{L^2} \\
&\geqslant c_5(g, \psi) \left\| T(\varphi)^{\frac{N+2}{2}} \right\|^2_{L^N}.
\end{aligned} \tag{5.7}$$

Combining this and (5.6), we obtain that

$$\begin{aligned}
c_5\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} &\leqslant \int_M \mathcal{B}^+_{\tau,\psi}\varphi^{N-1}T(\varphi)^{N+1}dv + \int_M \left(|\sigma|^2 + |LW|^2 + \pi^2\right)dv \\
&\leqslant \int_M \mathcal{B}^+_{\tau,\psi}\varphi^{N-2}T(\varphi)^{N+2}dv + 3c_3^2\kappa^{\frac{2}{N}} \quad \text{(by (5.5) and } \varphi \leqslant T(\varphi)) \\
&\leqslant \left(\max \mathcal{B}_{\tau,\psi}\right)\left(\int_M \varphi^{\frac{N(N+2)}{2}}dv\right)^{\frac{2(N-2)}{N(N+2)}}\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} + 3c_3^2\kappa^{\frac{2}{N}} \quad \text{(by Hölder inequality)} \\
&= (a\kappa)^{\frac{2(N-2)}{N(N+2)}}\left(\max \mathcal{B}_{\tau,\psi}\right)\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} + 3c_3^2\kappa^{\frac{2}{N}} \quad \left(\text{by } \int_M \varphi^{\frac{N(N+2)}{2}}dv = a\kappa\right).
\end{aligned} \tag{5.8}$$

It follows that

$$c_5\left(\int_M T(\varphi)^{\frac{N(N+2)}{2}}dv\right)^{\frac{2}{N}} \leqslant 6c_3^2k^{\frac{2}{N}},$$

provided $\kappa \leqslant a^{-1}\left(\frac{2\max \mathcal{B}^{V,+}_{\tau,\psi}}{c_5}\right)^{-\frac{N(N+2)}{2(N-2)}}$. On the other hand by (5.4), we obtain from the previous inequality that

$$c_5(a\kappa)^{\frac{2}{N}} \leqslant 6c_3^2\kappa^{\frac{2}{N}},$$

or equivalently,

$$a \leqslant \left(\frac{6c_3^2}{c_5}\right)^{\frac{N}{2}}. \tag{5.9}$$

However, it is worth recalling that c_3, c_5 only depend on (n, g, V, ψ, τ). Then (5.9) gives a contradiction if we choose a large enough such that the inverse inequality above holds.

Now we are are ready to prove the half-continuity of S. We have just proven that there exists $m \in M$ s.t. $0 < T(\varphi)(m) < \varphi(m)$. Then, since T is continuous on $\mathscr{C}$, there exists $\delta = \delta(\varphi) > 0$ small enough s.t.

$$0 < T(\omega)(m) < \omega(m), \quad \forall\omega \in B(\varphi,\delta)\cap\mathscr{C},$$

and hence from the fact that

$$-(S(\omega)(m) - \omega(m)) = \begin{cases} -\min\{T(\omega)(m), Q\} + \omega(m) & \text{if } \int_M \omega^{\frac{N(N+2)}{2}}dv \leqslant a\kappa \\ \omega(m) & \text{otherwise,}\end{cases}$$

we conclude that

$$-(S(\omega)(m) - \omega(m)) > 0$$

for all $\omega \in B(\varphi,\delta)\cap\mathscr{C}$, which implies the half-continuity of S at φ. Namely, this comes from the definition of half-continuity applied with $p(f) = -f(m)$ for all $f \in C^0$ (note that $p \in (C^0)^*$). The proof is completed. □

5.3 Nonuniqueness of Solutions

In this section, we will prove Theorem 5.1.1, which shows nonuniqueness of solutions to (3) in the scalar field case with freely specified mean curvature.

Proof of Theorem 5.1.1. It follows from Theorem 4.2.1 that for all $\epsilon > 0$ small enough the system (3) associated to $(g, \tau, \psi, V, \epsilon\sigma, \epsilon\pi)$ admits a solution $(\varphi_\epsilon, W_\epsilon)$ satisfying $\epsilon^{-\frac{2}{N+2}}\varphi_\epsilon \leqslant c$, for some constant $c > 0$ independent of ϵ. Thus, to show our theorem, it suffices to show that there exists a sequence $\{\epsilon_i\}$ converging to 0 s.t. the system (3) associated to $(g, \tau, \psi, V, \epsilon_i\sigma, \epsilon_i\pi)$ has a solution (φ_i, W_i) satisfying $\epsilon_i^{-\frac{2}{N+2}} \|\varphi_i\|_{L^\infty} \to \infty$.

In fact, since $\mathcal{B}_{\tau,\psi} > 0$ and $\pi \not\equiv 0$, we may let k be a fixed constant large enough s.t.

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M (k\pi)^{\frac{n+2}{2n}} \mathcal{B}_{\tau,\psi}^{\frac{3n-2}{4n}} dv > \int_M \left(\mathcal{R}_\psi^+\right)^{\frac{n+2}{4}} \mathcal{B}_{\tau,\psi}^{\frac{2-n}{4}} dv.$$

It follows by Theorem 1.3.2 that the Lichnerowicz equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + \psi\varphi = \mathcal{B}_{\tau,\psi}\varphi^{N-1} + \left(|k\sigma + LW|^2 + (k\pi)^2\right)\varphi^{-(N+1)}$$

has no solution for all $W \in C^1$, and hence the system (3) associated to $(g, \tau, \psi, V, k\sigma, k\pi)$ admits no solution.

Next we construct an operator T as follows. For each $(t, \varphi) \in [0, 1] \times L^\infty$, there exists a unique $W_\varphi \in W^{2,p}$ such that

$$-\frac{1}{2}L^*LW_\varphi = \frac{n-1}{n}\varphi^N d\tau - k\pi d\psi,$$

and arguing as in the proof of Theorem 1.2.3 in the case $\mathcal{Y}_g > 0$, there is then a unique $\theta \in W^{2,p}_+$ satisfying

$$\frac{4(n-1)}{n-2}\Delta\theta + \mathcal{R}_\psi\theta = t^{N+1}\mathcal{B}_{\tau,\psi}\varphi^{N-1} + \left(|k\sigma + LW_\varphi|^2 + (k\pi)^2\right)\theta^{-N-1}.$$

Therefore, we may define

$$T(t, \varphi) = \theta.$$

Similarly to the proof of Proposition 2.2.6, we obtain that T is a continuous compact.

Note that by the way of choosing k, the map $T(1, .)$ has no fixed point. It then follows from the Leray-Schauder fixed point theorem that there is a sequence $\{(t_i, \varphi_i)\}$ satisfying

$$\frac{4(n-1)}{n-2}\Delta\varphi_i + \mathcal{R}_\psi\varphi_i = t_i^{2N}\mathcal{B}_{\tau,\psi}\varphi_i^{N-1} + \left(|k\sigma + LW_i|^2 + (k\tau)^2\right)\varphi_i^{-N-1} \tag{5.10a}$$

$$-\frac{1}{2}L^*LW_i = \frac{n-1}{n}t_i^N\varphi_i^N d\tau - k\pi d\psi, \tag{5.10b}$$

where

$$\|\varphi_i\|_{L^\infty} \to \infty. \tag{5.11}$$

On the other hand, Premoselli showed in [Premoselli,] that if $(\varphi_\alpha, W_\alpha)$ is a solution to the Einstein equation (3) associated to the seed data $(V_\alpha, \tau_\alpha, \psi_\alpha, \pi_\alpha, \sigma_\alpha)$, which satisfies that

$$||V_\alpha - V_0||_{C^2} + ||\tau_\alpha - \tau_0||_{C^3} + ||\psi_\alpha - \psi_0||_{C^2} + ||\pi_\alpha - \pi_0||_{C^0} + ||\sigma_\alpha - \sigma_0||_{C^0} \ \overrightarrow{\alpha \to \infty} \ 0,$$

for some $(V_0, \tau_0, \psi_0, \pi_0, , \sigma_0) \in C^2 \times C^3 \times C^2 \times C^0 \times C^0$ with $\mathcal{B}_{\tau_0,\psi_0,V_0} > 0$, $\mathcal{R}_{\psi_0} > 0$ and $\pi_0 \not\equiv 0$, then (after passing to a subsequence) $(\varphi_\alpha, W_\alpha)$ converges to some solution (φ_0, W_0) of the Einstein equations (3) associated to the seed data $(V_0, \tau_0, \psi_0, \pi_0, \sigma_0)$. In particular, the sequence $\{\varphi_\alpha\}$ is uniformly bounded. Therefore, we must have in our current situation that t_i converges to 0.

Setting

$$\epsilon_i = t_i^{n(N+2)/2} k,$$

and by a straightforward calculation similar to the last step in the proof of Theorem 4.2.1, we obtain from (5.10) that the system (3) associated to $(g, \tau, \psi, V, \epsilon_i\sigma, \epsilon_i\tau)$ admits a solution $\left(\left(\frac{\epsilon_i}{k}\right)^{2/(N+2)} \varphi_i, \frac{\epsilon_i}{k} W_i\right)$. Combined with (5.11), this completes the proof.

$\square$

Chapter 6

On the Positive Mass Theorem for Asymptotically Hyperbolic Manifolds

6.1 Introduction

The mass of asymptotically hyperbolic manifolds introduced first by Wang [Wang, 2001] and Chruściel-Herzlich [Chruściel et Herzlich, 2003] plays an important role in the study of general relativity. Indeed, as explained in [Cortier *et al.*,], the mass of an asymptotically hyperbolic manifold is a vector that encompasses both the energy of the gravitational field and the location of the center of mass at least for asymptotically anti-de Sitter spacetimes.

A conjecture in this context, the so-called positive mass theorem (PMT), states that the mass vector is timelike future directed or zero for all complete asymptotically hyperbolic manifolds with scalar curvature greater than or equal to that of the hyperbolic space, $\mathrm{Scal} \geqslant -n(n-1)$. Further, the mass vector is zero only if the manifold is isometric to the hyperbolic space. This conjecture is known to hold in the spin case but, as for asymptotically Euclidean manifolds, the non-spin case is still open. For deeper discussions of these results, we refer the reader to Wang [Wang, 2001], Chruściel–Herzlich [Chruściel et Herzlich, 2003], Andersson–Cai–Galloway [Andersson *et al.*, 2008] where the PMT is proven in dimension less than 8 under a fairly restrictive assumption on the geometry at infinity of the metric, and to Dahl–Gicquaud–Sakovich [Dahl *et al.*, 2014].

Recently, E. Humbert and A. Hermann in [Humbert et Herman, 2014] have proven an interesting result on the PMT for closed Riemannian manifolds. They showed that if the PMT is true for all metrics on one closed simply connected non-spin manifold of dimension $n \geqslant 5$, then it holds on all closed manifolds of the same dimension. This provides a significant reduction; i.e., to show the PMT for closed manifolds of dimension $n \geqslant 5$, it suffices to study a single closed simply connected non-spin manifold of the same dimension. It is then natural to ask if a similar result exists for asymptotically hyperbolic manifolds.

In this chapter, we are interested in this question. We show that if there exists one manifold M of dimension $n \geqslant 5$ such that all asymptotically hyperbolic metric with scalar curvature greater than or equal to $-n(n-1)$ on M have timelike or null future oriented mass vector (this is what we will call the weak PMT), then this property holds for all asymptotically hyperbolic manifold of the same dimension. The proof relies on surgery theory; see e.g. [Kosinski, 1993] for an introduction.

It should be noted that, since all 3-manifolds are spin, the positive mass theorem holds for all asymptotically hyperbolic 3-manifolds. Still, the case of non-spin 4-dimensional manifolds remains open in full generality. It is expected that the arguments developed in [Sakovich, 2015] can be generalized to prove the positive mass theorem for all asymptotically hyperbolic manifolds of dimension less than eight.

We show that a certain rigidity statement of the positive mass theorem also holds. Namely, if the mass vector of some asymptotically hyperbolic metric is zero, then the metric is isometric to the hyperbolic metric. The idea of the proof uses an argument taken from [Dahl *et al.*, 2014] to prove that metrics for which the mass vector is zero are static metrics. Then adapting a result of [Qing, 2003] which shows that the only complete static metric having the round sphere as conformal infinity is the hyperbolic space, we deduce the rigidity of the positive mass theorem. Note however that the argument is not robust enough yet to address the possibility of a lightlike mass vector.

The outline of this chapter is as follows. In Section 6.2, we introduce the basic definitions that will be used all along the chapter. Section 6.3 contains the first result of the chapter, namely that the category of manifolds for which the weak PMT holds is stable by surgeries of codimension greater than 2. Section 6.4 contains the proof of the rigidity statement of the positive mass theorem assuming only the weak form of the PMT.

6.2 Preliminaries

We denote by $(\mathbb{H}^n, b)$ the n−dimensional hyperbolic space, with $n \geqslant 3$. We fix a point in $\mathbb{H}^n$ as an origin. Then, in geodesic normal coordinates at this point, the hyperbolic metric reads $b = dr^2 + \sinh^2 r\sigma$, where σ is the standard round metric on $\mathbb{S}^{n-1}$ and r is the distance from the origin. We also denote the open ball of radius R centered at the origin and its closure by B_R and $\overline{B}_R$ respectively.

Two other models of the hyperbolic space will be used:

- *The hyperboloidal model*: The hyperbolic space can be embedded isometrically into Minkowski space $\mathbb{R}^{n,1}$ as the hypersurface

$$H^n := \{(x^0, x^1, ..., x^n) \in \mathbb{R}^{n+1}, -(x^0)^2 + (x^1)^2 + ... + (x^n)^2 = -1, x^0 > 0\}.$$

- *The ball model*: Another useful model of the hyperbolic space is the Poincaré ball model. The hyperbolic space $(\mathbb{H}^n, b)$ can be viewed as the unit ball $B_1(0)$ of $\mathbb{R}^n$ endowed with the metric

$$b = \rho^{-2}\delta,$$

where δ is the Euclidean metric and $\rho := \frac{1-|x|^2}{2}$ is the standard defining function for the sphere $S_1(0)$.

Given some $\alpha \in (0,1)$ and $\tau \in (\frac{n}{2}, n)$, a Riemannian manifold (M,g) is called $C^{2,\alpha}_{\tau}$-asymptotically hyperbolic if there is a compact subset $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \longrightarrow \mathbb{H}^n \setminus \overline{B}_R$ for which $\Phi_* g$ and b are uniformly equivalent on $\mathbb{H}^n \setminus \overline{B}_R$ and

$$\begin{aligned} \int_{\mathbb{H}^n \setminus B_R} \left| \mathrm{Scal}^g + n(n-1) \right| \cosh(r)\, d\mu^b &< \infty, \\ \|e\|_{C^{2,\alpha}_{\tau}(\mathbb{H}^n \setminus B_R, S^2M)} := \sup_{x \in \mathbb{H}^n \setminus B_R} e^{\tau r(x)} \|e\|_{C^{2,\alpha}(B_1(x), S^2M)} &< \infty, \end{aligned} \tag{6.1}$$

where $e := \Phi_* g - b$. The diffeomorphism Φ is also called a chart, or a set of coordinates, at infinity.

Now we follow the work of Chruściel and Herzlich, [Chruściel et Herzlich, 2003] and [Herzlich, 2005], to define the mass of an asymptotically hyperbolic manifold (M,g). Let $\mathcal{N} := \left\{ V \in C^{\infty}(\mathbb{H}^n) \mid \mathrm{Hess}^b\, V = Vb \right\}$. This vector space gets identified with restrictions to the hyperboloid H^n of linear forms on $\mathbb{R}^{n,1}$. In particular, it has a basis consisting of the functions

$$V_{(0)} = \cosh(r),\ V_{(1)} = x^1 \sinh(r),\ V_{(n)} = x^n \sinh(r),$$

where the functions $x^1, ..., x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}. The vector space $\mathcal{N}$ is equipped with a Lorentzian inner product η characterized by the condition that the above basis is orthonormal: $\eta(V_{(0)}, V_{(0)}) = 1$, and $\eta(V_{(i)}, V_{(i)}) = -1$ for $i = 1, ..., n$. We give $\mathcal{N}$ a time orientation by specifying the vector $V_{(0)}$ to be future directed. The subset $\mathcal{N}^+$ of functions bounded from below by a positive constant then coincides with the interior of the future lightcone.

The linear functional H_{Φ} on $\mathcal{N}$ is defined by

$$H_{\Phi}(V) = H^g_{\Phi}(V) := \lim_{r \to \infty} \int_{S_r} \left(V \left(\mathrm{div}^b\, e - d\, \mathrm{tr}^b\, e \right) + \left(\mathrm{tr}^b\, e \right) dV - e\left(\nabla^b V, . \right) \right) (\nu_r) d\mu^b \tag{6.2}$$

is called the *mass functional* of (M,g) with respect to Φ. [Chruściel et Herzlich, 2003, Proposition 2.2] tells us that the limit involved in the definition of H_{Φ} exists and is finite when the decay conditions (6.1) are satisfied for some $\tau > n/2$. Since H_{Φ} is a linear form on the dual of $\mathbb{R}^{n,1}$ it gets identified with a unique element of $\mathbb{R}^{n,1}$.

As for asymptotically Euclidean manifolds, we introduce the following concept.

Definition 6.2.1. *Let M be an open manifold and Φ be a diffeomorphism from the exterior of a compact $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$. We say that (M,Φ) satisfies the weak positive mass theorem if for any metric g on M which is $C^{2,\alpha}_{\tau}$–asymptotically hyperbolic w.r.t. the diffeomorphism Φ, with $\alpha \in (0,1)$ and $\tau > n/2$, and such that $\mathrm{Scal}_g = -n(n-1) + O(e^{-(n-1+\xi)r})$ for some small constant $\xi > 0$, we have*

$$H_{\Phi}(V) \geqslant 0, \tag{6.3}$$

for all non-zero $V \geqslant 0$. In addition if the equality in (6.3) is achieved only for g being isometric to b, we say that (M,Φ) satisfies the strong PMT.

6.3 The main results and proofs

In this section, we shall obtain a result analogous to [Humbert et Herman, 2014, Theorem 8.5] for asymptotically hyperbolic manifolds, which allows us to prove that it suffices to consider the PMT on a single simply connected non-spin manifold rather than a general one. The key point in the argument of Humbert-Hermann is that the property of satisfying the PMT is not affected by doing a finite sequence of surgeries. To obtain this for closed manifolds, they characterize the Green's functions associated to certain second order elliptic operators. No such formulation is known for the mass of an asymptotically hyperbolic manifold so we need to modify their argument. This will be presented in the following propositions.

Proposition 6.3.1. *Let (M^n, g) be a $C^{2,\alpha}_\tau$-asymptotically hyperbolic manifold of dimension n for some $n \geqslant 3$, with $\alpha \in (0,1)$ and $\tau \in (n/2, n)$. Assume that g does not satisfy the weak PMT. Then given any compact subset K of M, there exists a metric $\widetilde{g}$ on M such that*

$$\mathrm{Scal}^{\widetilde{g}} \geqslant -n(n-1) \ \text{ on } M, \quad \mathrm{Scal}^{\widetilde{g}} > -n(n-1) \ \text{ on } K$$

and $\widetilde{g}$ does not satisfy the PMT.

Proof. Since g does not satisfy the weak PMT, there exists a function $V \in \mathcal{N}^+$ such that $H^g_\Phi(V) < 0$. We first modify g so that it has nonpositive scalar curvature.

Denote by U the open subset $U := \{m \in M : \ \mathrm{Scal}^g(m) > 0\}$. Applying [Lohkamp, 1999, Theorem 1] to U and the function $f = \min\{0, \mathrm{Scal}^g\}$, and for $\epsilon > 0$ small enough, we obtan a metric g_ϵ on M s.t.

$$g \equiv g_\epsilon \ \text{ on } M \setminus U_\epsilon \text{ and } \ -n(n-1) \leqslant \mathrm{Scal}^{g_\epsilon} \leqslant f \ \text{ on } U_\epsilon,$$

where U_ϵ is the $\epsilon-$neighborhood of U w.r.t. g. In particular, we have that

$$-n(n-1) \leqslant \mathrm{Scal}^{g_\epsilon} \leqslant 0. \tag{6.4}$$

Since $\mathrm{Scal}^g \to -n(n-1)$ at infinity, U must be bounded, so g_ϵ coincide with g outside a compact set. In particular, we conclude that $H^{g_\epsilon}_\Phi(V) < 0$. As a consequence, we may assume further that $-n(n-1) \leqslant \mathrm{Scal}^g \leqslant 0$ without loss of generality.

Now let $\chi \geqslant 0$ be a smooth nonnegative function s.t. χ has compact support in M and $\chi \equiv 1$ on K. Next we define $F : \mathbb{R} \times C^{2,\alpha}_\tau \to C^{0,\alpha}_\tau$ by

$$F(\lambda, u) = \frac{4(n-1)}{n-2}\Delta(u+1) + \mathrm{Scal}^g(u+1) - (\lambda\chi + \mathrm{Scal}^g)(u+1)^{N-1}, \tag{6.5}$$

where $N = 2n/(n-2)$. Obviously, we have that $F(0,0) = 0$ and standard computation shows that the Fréchet derivative of F w.r.t. u at $(0,0)$ is given by

$$\mathcal{D}F_0(0)(v) = \frac{4(n-1)}{n-2}\Delta v - (N-2)\mathrm{Scal}^g v. \tag{6.6}$$

By [Lee, 2006, Theorem C], $\mathcal{D}F_0(0) : C^{2,\alpha}_\tau \to C^{0,\alpha}_\tau$ is Fredholm with zero index provided $\tau \in (\frac{n}{2}, n)$ (in fact, it is true for all $\tau \in (-1, n)$). Moreover, since $\mathrm{Scal}^g \leqslant 0$ and $\mathrm{Scal}^g \not\equiv 0$, $\mathcal{D}F_0(0)(.)$ has a trivial kernel. Namely, $\mathcal{D}F_0(0)(.)$ is an isomorphism by the maximum principle.

The implicit function theorem then implies that there exists a sequence $\{\lambda_k, u_k\}$ converging to $(0,0)$ (with $\lambda_k > 0$ and $u_k \not\equiv 0$) satisfying

$$\frac{4(n-1)}{n-2}\Delta(u_k+1) + \mathrm{Scal}^g(u_k+1) = (\lambda_k\chi + \mathrm{Scal}^g)\,(u_k+1)^{N-1}.$$

Therefore, taking $g_k = (u_k+1)^{N-2}g$, it follows that

$$\mathrm{Scal}^{g_k} := \lambda_k\chi + \mathrm{Scal}^g$$

satisfies the first two conditions of our assertion. On the other hand, using the formulas from [Herzlich, 2005, page 114] we have that with R large enough

$$\begin{aligned}\left|H_\Phi^{g_k}(V) - H_\Phi^{g}(V)\right| = &\int_{S_R}\left(V\Big[\,\mathrm{div}^b(e_k-e) - d\ \mathrm{tr}^b(e_k-e)\Big] + \mathrm{tr}^b(e_k-e)dV - (e_k-e)\left(\nabla^b V,.\right)\right)(\nu_R)d\mu^b\\ &+\int_{\mathbb{H}^n\setminus B_R}\Big(V\,(\mathrm{Scal}^{g_k}-\mathrm{Scal}^g) + Q(e_k,V) - Q(e,V)\Big)d\mu^b\end{aligned}$$

(see also [Dahl *et al.*, 2014, Proposition B.1]). Since $\mathrm{Scal}^{g_k} = \mathrm{Scal}^g$ outside the (compact) support of χ and u_k tends to 0 in $C^{2,\alpha}_\tau$ as $k \to \infty$, and since $Q(e,v)$ is quadratic in e and ∇e, we may take k large enough s.t. all these terms are as small as we want. This means that $H_\Phi^{g_k}(V) < 0$. Hence $\widetilde{g} = g_k$ is our desired metric. □

Proposition 6.3.2. *Let (N^n, g_0) be an asymptotically hyperbolic manifold of dimension n for some $n \geqslant 3$, and let M^n be obtained from N by a surgery of codimension $q \geqslant 3$. Assume that g_0 does not satisfy the weak PMT. Then there exists an asymptotically hyperbolic metric g on M such that* $\mathrm{Scal}^g \geqslant -n(n-1)$ *on M and g does not satisfy the weak PMT.*

Proof. Suppose that S^p is a given embedded sphere in N of codimension $q = n-p \geqslant 3$, with trivial normal bundle and on which we are going to do a surgery. Our situation is similar to the well-known surgery theorem proven by Gromov-Lawson [Gromov et Lawson, 1980, Theorem A] and Rosenberg-Stolz [Rosenberg et Stolz, 2001, Theorem 3.1], which ensures that the positivity of scalar curvature on a given closed Riemannian manifold can be preserved after doing a surgery. Thus, our next arguments follow the proof of [Rosenberg et Stolz, 2001, Theorem 3.1], which is sketched out as follows.

Let K be an arbitrary compact set containing S^p in its interior. By Proposition 6.3.1, we may construct a metric g_1 on N s.t.

$$\mathrm{Scal}^{g_1} \geqslant -n(n-1)\ \text{ on } N,\quad \mathrm{Scal}^{g_1} > -n(n-1)\ \text{ on } K \tag{6.7}$$

and g_1 violates the weak PMT. The proposition will follow if we can construct a metric g on M which coincides with g_1 outside of K and keeps the second property in (6.7) after doing the surgery.

Set $q = n-p$. By the exponential map we can specify a tubular neighborhood $S^p \times D^q(\bar{s})$ of S^p for some $\bar{s} > 0$ such that the radial coordinate s on $D^q(\bar{s})$ measures distances from $S^p \times \{0\}$ w.r.t. the metric g_1. All of our work will then take place in this neighborhood. Upon reducing $\bar{s}$, we may further assume that $S^p \times D^q(\bar{s}) \subset K$ without loss of generality.

We next consider a hypersurface T of the Riemannian product $N \times \mathbb{R}$, defined by

$$T = \{(y, x, t) \in S^p \times D^q(\bar{s}) \times \mathbb{R} : \ (t, s = |x|) \in \gamma\},$$

where γ is a smooth curve in $t - s$ plane, satisfying the following properties:

1. γ lies in the region $0 < s \leqslant \bar{s}$ of $t - s$ plane,
2. γ begins at one end with a vertical line segment $t = 0, \ s_1 \leqslant s \leqslant \bar{s}$,
3. γ ends with a horizontal line segment $s = s_\infty$, with s_∞ "small",
4. In the region $s_\infty < s < s_1$, γ is the graph of a function $s = f(t)$ which is decreasing and (weakly) concave upward.
5. γ is chosen so that the scalar curvature of T is strictly greater than $-n(n-1)$.

One may wonder whether T is well-defined. To ensure this, we need to show that such a smooth curve γ exists. In fact, there is no problem with the first four conditions. The last one will be obtained by arguments similar to [Rosenberg et Stolz, 2001, Theorem 3.1]. We first note that since T is a hypersurface of $N \times \mathbb{R}$, it follows from the Gauss equation that

$$\mathrm{Scal}_T = \mathrm{Scal}^{g_1} + O(1)\sin^2\theta + (q-1)(q-2)\frac{\sin^2\theta}{s^2} - (q-1)\frac{\kappa\sin^2\theta}{s} - O(s)(q-1)\kappa\sin\theta, \quad (6.8)$$

where Scal_T is the scalar curvature of T, κ is the curvature of γ (as a curve in the Euclidean plane), and θ is the angle between γ and a vertical line. See [Walsh, 2011, Appendix] for complete detail.

Since $\mathrm{Scal}^{g_1} > -n(n-1)$ on $S^p \times D^q(\bar{r}) \subset K$, we can choose a constant $\kappa_0 > 0$ such that $\mathrm{Scal}^{g_1} + n(n-1) > (q-1)\kappa_0$ on $S^p \times D^q(\bar{s})$. We then get from (6.8) that to satisfy the last condition, it is sufficient to find γ satisfying

$$\left(1 + c_1 s^2\right)\kappa \leqslant (q-2)\frac{\sin\theta}{s} + \kappa_0\frac{s}{\sin\theta} - c_2 s\sin\theta, \quad (6.9)$$

where the constants c_1, c_2 come from the $O(1)$ term and the $O(s)$ term in (6.8).

The situation arising here coincides with the one in the proof of [Rosenberg et Stolz, 2001, Theorem 3.1] so the interested reader can consult there the details of the proof.

We now show how the $t - s$ plane works in the construction of the metric g. By the second condition on Γ the metric on T is isometric to a portion of N in a collar of one component of ∂T. We can then glue T onto $N \setminus (S^p \times D^q(\bar{s}))$ to obtain a Riemannian manifold (N', g') with a single boundary component $S^p \times S^{q-1}(s_\infty)$ such that

- $\mathrm{Scal}^{g'} \geqslant -n(n-1)$ on N' ,
- g' is the product metric in a collar neighborhood of the boundary.

Note that since $q-1\geqslant 2$ and $s_\infty>0$ is very small, [Gromov et Lawson, 1980, Lemma 2] shows that there exists an isotopy ds_l^2 (i.e., a 1-parameter family of metrics with scalar curvature being strictly greater than $-n(n-1)$), $0\leqslant l\leqslant 1$, between the current metric on $S^p\times S^{q-1}(s_\infty)$ and the product of two standard spheres $S^p\times S^{q-1}(s_\infty)$. Then, if we consider the metric $ds_{l/a}^2+dl^2$ on $W=S^p\times S^{q-1}(s_\infty)\times[0,a]$, we will see that the associated scalar curvature Scal_W on this cylinder is greater than $-n(n-1)$, provided a is large enough. In fact, a calculation (see [Gromov et Lawson, 1980, Lemma 3]) shows that for a given $(x,l)\in\left(S^p\times S^{q-1}(s_\infty)\right)\times[0,a]$,

$$\mathrm{Scal}_W(x,l)=\kappa_{l/a}(x)+O(1/a),$$

where $\kappa_{l/a}$ is the scalar curvature of $S^p\times S^{q-1}(s_\infty)$ for the metric $ds_{l/a}^2$.

We now glue this cylinder onto N' to get a Riemannian manifold (N'',g''), whose boundary $S^p\times S^{q-1}(s_\infty)$ is the Riemannian product of two standard spheres, such that:

- $\mathrm{Scal}^{g''}\geqslant -n(n-1)$ on N'',
- g'' is the product metric in a collar neighborhood of the boundary.

Finally, we continue to glue onto N'' a Riemannian product $D^{p+1}\times S^{q-1}(s_\infty)$, where the disk D^{p+1} has a metric which is a Riemannian product $S^p\times[0,b]$ in a neighborhood of the boundary. The endproduct of the construction is our desired metric. □

We are now ready to state and prove our main theorem.

Theorem 6.3.3. *If the weak PMT is true on a simply connected non-spin asymptotically hyperbolic manifold (M,Φ) of dimension $n\geqslant 5$, then so it is on all asymptotically hyperbolic manifolds of the same dimension.*

Proof. We first need to construct a compact manifold $\widehat{M}$ from any (M,Φ) as follows. We recall that for $\epsilon>0$ small, $\Phi : M\setminus K\to\mathbb{H}^n\setminus B_R=B_1(0)\setminus\bar{B}_{1-\epsilon}(0)$ is a diffeomorphism. Then we may let $\sim$ be the equivalence relation in $M\coprod B_{\frac{1}{1-\epsilon}}(0)$ given by

$$p\in M\ \sim\ q\in B_{\frac{1}{1-\epsilon}}\ \text{ if }\ \Phi(p)=\frac{q}{|q|^2}.$$

Now the compact manifold $\widehat{M}$ is defined by

$$\widehat{M}:=\left(M\coprod B_{\frac{1}{1-\epsilon}}(0)\right)/\sim.$$

This construction corresponds to closing the end of M. Next, our assertion will follow the proof of [Humbert et Herman, 2014, Theorem 8.5]. Assume that M satisfies the weak PMT. We will prove that $M\sharp(-\widehat{M})$ satisfies the weak PMT by contradiction. In fact, assume that it is not true, then $M\coprod\left(-\widehat{M}\right)\sharp\widehat{M}$ does not satisfy the weak PMT. Since $M\sharp(-\widehat{M})\sharp\widehat{M}$ can be obtained from $M\coprod\left(-\widehat{M}\right)\sharp\widehat{M}$ by connected sum (i.e., a surgery of dimension 0), it follows from Proposition 6.3.2 that $M\sharp\widehat{M}\sharp(-\widehat{M})$ does not satisfy the weak PMT. On the other hand, since $M\sharp\widehat{M}\sharp(-\widehat{M})$ is cobordant to M and both are non-spin, M can be obtained from $M\sharp\widehat{M}\sharp(-\widehat{M})$ by finitely many surgeries of dimension $k=\{0,...,n-3\}$.

Therefore, we also conclude from Proposition 6.3.2 that M does not satisfy the weak PMT, which is a contradiction.

Now let (N, Ψ) be any given asymptotically hyperbolic manifold of the same dimension. We will show that N satisfies the weak PMT. We argue again by contradiction. Assume that our claim is not true. Then $N \coprod (-\widehat{N})$ does not satisfy the weak PMT. Moreover, since $N\sharp(-\widehat{N})$ can be obtained from $N \coprod (-\widehat{N})$ by a surgery of dimension 0, it follows from Proposition 6.3.2 that $N\sharp(-\widehat{N})$ does not satisfy the weak PMT. However, since $N\sharp(-\widehat{N})$ is cobordant to $M\sharp(-\widehat{M})$ (they are both cobordant to a disk) and both are non-spin, we have that $M\sharp(-\widehat{M})$ can be obtained from $N\sharp(-\widehat{N})$ by finitely many surgeries of dimension $k = \{0, ..., n-3\}$, and then by Proposition 6.3.2 we again conclude that $M\sharp(-\widehat{M})$ does not satisfy the weak PMT, which is a contradiction. The proof of Theorem 6.3.3 is completed. □

6.4 On the rigidity case of the positive mass theorem

The aim of this section is to prove the following result:

Theorem 6.4.1. *Assuming that the positive mass theorem for asymptotically Euclidean manifolds is true, let M be an open manifold and Φ be a diffeomorphism from the exterior of a compact $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$. Let g be a Riemannian metric on M such that $e := \Phi_* g - b \in C^{4,\alpha}_\tau$. If (M, Φ) satisfies the weak PMT and g has vanishing mass; i.e., if $H^g_\Phi(V) = 0$ for some $V \in \mathcal{N}^+$, then g is isometric to the hyperbolic metric.*

Before proving Theorem 6.4.1, we need to establish some preliminary results. In order to keep notations simple, we will not indicate pullbacks and push forwards with respect to the diffeomorphism Φ. Note also that, by changing the coordinate system at infinity, we can assume, without loss of generality, that $V = \cosh(r)$.

Lemma 6.4.2. *Assume that (M, g) is a $C^{2,\alpha}_\tau$–asymptotically hyperbolic manifold. Then for any $\varphi \in C^2$, $\varphi = \varphi(r)$, we have*

$$\left|\Delta^b \varphi - \Delta\varphi\right| \leqslant O(e^{-\tau r}) \max\left\{\left|\nabla^b \varphi\right|_b, \varphi', \varphi''\right\}, \tag{6.10}$$

where a prime denotes a derivative with respect to r.

Proof. We have that

$$\begin{aligned}\left|\Delta^b \varphi - \Delta\varphi\right| \leqslant &\left|g^{-1}\right|_b \left|\mathrm{Hess}^g \varphi - \mathrm{Hess}^b \varphi\right|_b \\ &+ \left|g^{-1} - b^{-1}\right|_b \left|\mathrm{Hess}^b \varphi\right|_b .\end{aligned} \tag{6.11}$$

By straightforward computation, we get that

$$\mathrm{Hess}^g_{ij} \varphi - \mathrm{Hess}^b_{ij} \varphi = \left(\Gamma^k_{ij}(b) - \Gamma^k_{ij}(g)\right) \partial_k \varphi$$

and since $e \in C^{2,\alpha}_\tau$, we obtain that

$$\left|\mathrm{Hess}^g \varphi - \mathrm{Hess}^b \varphi\right|_b \leqslant |\Gamma(b) - \Gamma(g)|_b \left|\nabla^b \varphi\right|_b \leqslant O(e^{-\tau r}) \left|\nabla^b \varphi\right|_b . \tag{6.12}$$

For estimating the second term in the right hand side of (6.11), we let (μ, ν) indicate directions tangential to the spheres of radius r, so the associated second fundamental form is then written as

$$S_{\mu\nu} = \coth r b_{\mu\nu}.$$

Since

$$\mathrm{Hess}^b_{rr}(\varphi - 1) = \varphi'', \qquad \mathrm{Hess}^b_{r\mu}(\varphi - 1) = 0 \quad \text{and} \quad \mathrm{Hess}^b_{\mu\nu}(\varphi - 1) = S_{\mu\nu}\varphi',$$

we have that

$$\begin{aligned} \left|g^{-1} - b^{-1}\right|_g \left|\mathrm{Hess}^b(\varphi - 1)\right|_b &= \left|g^{-1} - b^{-1}\right|_g \left((\varphi'')^2 + |S|^2 (\varphi')^2\right)^{\frac{1}{2}} \\ &\leqslant O(e^{\tau r}) \left((\varphi'')^2 + (\varphi')^2\right)^{\frac{1}{2}} \qquad (\text{since } e \in C^{2,\alpha}_\tau) \end{aligned} \tag{6.13}$$

Finally, combining (6.11)-(6.13) we obtain (6.10). □

Lemma 6.4.3. *Assume that (M, g) be a $C^{2,\alpha}_\tau$−asymptotically hyperbolic. Assume further that* $\mathrm{Scal}^g \geqslant -n(n-1)$. *Then there exists $r_0 > 0$ s.t. for all $\epsilon \in (0, 1)$,*

$$\varphi_\epsilon := 1 - \epsilon e^{-nr} - \epsilon e^{-(n+1)r}$$

is a supersolution to the Yamabe equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}^g\varphi = -n(n-1)\varphi^{N-1}, \tag{6.14}$$

on $M \setminus B_{r_0}$.

Proof. We need to prove that there exists $r_0 > 0$ s.t.

$$\frac{4(n-1)}{(n-2)}\Delta\varphi_\epsilon + \mathrm{Scal}^g\varphi_\epsilon + n(n-1)\varphi_\epsilon^{N-1} \geqslant 0 \qquad \text{on } M \setminus B_{r_0}, \tag{6.15}$$

for all $\epsilon \in (0, 1)$. Note that Inequality (6.15) may be rewritten as

$$\frac{4(n-1)}{(n-2)}\Delta\varphi_\epsilon + (\mathrm{Scal}^g + n(n-1)) + \mathrm{Scal}^g (\varphi_\epsilon - 1) + n(n-1)\left(\varphi_\epsilon^{N-1} - 1\right) \geqslant 0.$$

Since $\mathrm{Scal}^g \geqslant -n(n-1)$ and since $\varphi_\epsilon^{N-1} - 1 \geqslant (N-1)(\varphi_\epsilon - 1)$, the previous inequality will be satisfied provided that

$$\Delta(\varphi_\epsilon - 1) + n(\varphi_\epsilon - 1) \geqslant 0. \tag{6.16}$$

Now we have by straightforward computation that

$$\begin{aligned} \Delta^b(\varphi_\epsilon - 1) + n(\varphi_\epsilon - 1) &= \epsilon n(n-1)(1 - \coth r)\, e^{-nr} \\ &\quad + \epsilon\left(n + 2 - \left(n^2 - 1\right)(\coth r - 1)\right) e^{-(n+1)r} \\ &\geqslant \epsilon e^{-(n+1)r}, \end{aligned} \tag{6.17}$$

where the last inequality holds since $\coth r = 1 + O(e^{-2r})$ and r is large enough independently of ϵ.

On the other hand, by Lemma 6.4.2, we get that

$$\begin{aligned} \left|\Delta^b \varphi_\epsilon - \Delta \varphi_\epsilon\right| &\leqslant O(e^{-\tau r}) \max \left\{ \left|\nabla^b \varphi_\epsilon\right|_b, \varphi'_\epsilon, \varphi''_\epsilon \right\} \\ &\leqslant \epsilon O\left(e^{-(n+\tau)r}\right). \end{aligned} \tag{6.18}$$

Finally, since $\tau > \frac{n}{2} > 1$, we conclude from (6.17)-(6.18) that there exists $r_0 > 0$ large enough s.t. (6.16) is satisfied on $M \setminus B_{r_0}$ for all $\epsilon \in (0,1)$. □

Lemma 6.4.4. *Assume that (M, g) be a $C^{2,\alpha}_\tau$−asymptotically hyperbolic. There exists a unique solution V^g to the equation*

$$\Delta^g V^g = nV^g \tag{6.19}$$

such that $|V^g - V| = O(e^{-(\tau-1)r})$ at infinity.

Proof. We have from Lemma 6.4.2 that

$$\Delta^g V = \Delta^b V + O(e^{-\tau r})e^r = nV + f(r) + O(e^{(1-\tau)r}).$$

This equation may be rewritten as

$$\Delta^g V + nV = \theta, \tag{6.20}$$

with $\theta \in C^{0,\alpha}_{\tau-1}$. Here we assume that V has been extended in some irrelevant way to the whole of M. On the other hand, since $\tau - 1 \in (-1, n)$, we have from [Lee, 2006, Theorem C] that there exists a unique $u \in C^{2,\alpha}_{\tau-1}$ s.t.

$$\Delta^g u + nu = \theta. \tag{6.21}$$

Now, we set $V^g = V - u$. It is not difficult to see from (6.20) and (6.21) that V^g is our desired function. The proof is complete. □

We are now ready to prove the main result of this section.

Proof of Theorem 6.4.1. Assume that there exists non-zero function $V \in \mathcal{N}^+$ s.t. $H_\Phi(V) = 0$. Composing the diffeomorphism Φ with a Lorentz transformation, we may assume without loss of generality that $V = \lambda \cosh(r(x))$ for some constant $\lambda > 0$. H_Φ being linear in V we can also assume that $V = \cosh(r)$. We divide the proof into three step.

Step 1. $\mathrm{Scal}^g = -n(n-1)$:

By [Gicquaud, 2010], there exists a unique solution $\varphi > 0$ to the Yamabe equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}^g\varphi = -n(n-1)\varphi^{N-1}, \tag{6.22}$$

satisfying that $\varphi - 1 \in C^{2,\alpha}_\tau$. This means that $\hat{g} = \varphi^{N-2}g$ has constant scalar curvature $\mathrm{Scal}^{\hat{g}} = -n(n-1)$. To show Step 1, it suffices to show that $\varphi \equiv 1$. We argue by contradiction. Assume that $\mathrm{Scal}^g \geqslant -n(n-1)$ and $\mathrm{Scal}^g \not\equiv -n(n-1)$. First we will prove that there exists $\epsilon_0 > 0$ satisfying

$$\varphi \leqslant \varphi_{\epsilon_0} = 1 - \epsilon_0 e^{nr} - \epsilon_0 e^{(n+1)r} \tag{6.23}$$

near infinity. In fact, since 1 is a supersolution to (6.22), we obtain $\varphi \leqslant 1$ on M, and then by the strong maximum principle $\varphi < 1$. Now by Lemma 6.4.3, there exists $R > 0$ large enough s.t. $\varphi_\epsilon := 1 - \epsilon_0 e^{nr} - \epsilon e^{(n+1)r}$ is a supersolution to (6.22) on $M \setminus B_R$ for all $\epsilon > 0$ small enough. Therefore, taking $\epsilon_0 \in (0,1)$ s.t.

$$\max_{\partial B_R} \varphi \leqslant 1 - \epsilon_0 e^{-nR} - \epsilon_0 e^{-(n+1)R},$$

we obtain by the maximum principle that $\varphi \leqslant \varphi_{\epsilon_0}$ as claimed.

Now, by definition we have that

$$H_\Phi^{\hat{g}}(V) = \lim_{r\to\infty} \int_{S_r} \left(V\left(\operatorname{div}^b \hat{e} - d\operatorname{tr}^b \hat{e}\right) + \left(\operatorname{tr}^b \hat{e}\right) dV - \hat{e}\left(\nabla^b V, .\right)\right)(\nu_r) d\mu^b,$$

where

$$\begin{aligned}
\hat{e} := \Phi_* \hat{g} - b &= \Phi_*\left(\varphi^{N-2} g\right) - b \\
&= \varphi^{N-2}(b+e) - b \\
&= e + \left(\varphi^{N-2} - 1\right) b + \left(\varphi^{N-2} - 1\right) e.
\end{aligned}$$

Since $e \in C^{2,\alpha}_\tau(M, S^2M)$ and $\operatorname{Scal}^g = -n(n-1) + O(e^{-(n-1+\xi)r})$ for some $\xi > 0$, we get that $\varphi - 1 \in C^{2,\alpha}_\tau(M, S^2M)$. Then the term $\left(\varphi^{N-2} - 1\right) e \in C^{2,\alpha}_{2\tau}$ gives no contribution in the limit $H_\Phi^{\hat{g}}(V)$. This means that

$$\begin{aligned}
H_\Phi^{\hat{g}}(V) &= \lim_{r\to\infty} \int_{S_r} \Big\{ V\left[\operatorname{div}^b\left(e + \left(\varphi^{N-2}-1\right)b\right) - d\operatorname{tr}^b\left(e + \left(\varphi^{N-2}-1\right)b\right)\right] \\
&\qquad + \left(\operatorname{tr}^b\left(e + \left(\varphi^{N-2}-1\right)b\right)\right) dV - \left(e + \left(\varphi^{N-2}-1\right)b\right)\left(\nabla^b V, .\right)\Big\}(\nu_r) d\mu^b \\
&= \lim_{r\to\infty} \int_{S_r} \left(V\left(\operatorname{div}^b e - d\operatorname{tr}^b e\right) + \left(\operatorname{tr}^b e\right) dV - e\left(\nabla^b V, .\right)\right)(\nu_r) d\mu^b \\
&\quad + (n-1) \lim_{r\to\infty} \int_{S_r} \left[\left(\varphi^{N-2}-1\right) dV(\nu_r) - V\nabla^b\left(\varphi^{N-2}-1\right)(\nu_r)\right] d\mu^b \\
&= \underbrace{H_\Phi^g(V)}_{=0} - (n-1) \int_{S_r} V^2 \nabla^b\left(\frac{\varphi^{N-2}-1}{V}\right)(\nu_r) d\mu^b
\end{aligned} \tag{6.24}$$

Since $V = \cosh r$, we have by straightforward computation that

$$\begin{aligned}
\lim_{r\to\infty} \int_{S_r} V^2 \nabla^b\left(\frac{\varphi^{N-2}-1}{V}\right)(\nu_r) d\mu^b &= \lim_{r\to\infty} \left\{ r^{-1} \int_r^{2r} \left[\int_{S_t} V^2 \nabla^b\left(\frac{\varphi^{N-2}-1}{V}\right)(\nu_r) d\mu^b\right] dt \right\} \\
&= \lim_{r\to\infty} \left\{ r^{-1} \left[\int_{S_{2r}} V\left(\varphi^{N-2}-1\right) d\mu^b - \int_{S_r} V\left(\varphi^{N-2}-1\right) d\mu^b\right]\right\} \\
&\qquad - \lim_{r\to\infty} r^{-1} \int_{B_{2r}\setminus B_r} \frac{\varphi^{N-2}-1}{V} \frac{\nabla_r\left(V^2 (\sinh r)^{n-1}\right)}{(\sinh r)^{n-1}} d\mu^b \\
&= -\lim_{r\to\infty} r^{-1} \int_{B_{2r}\setminus B_r} \frac{\varphi^{N-2}-1}{V} \frac{\nabla_r\left(V^2 (\sinh r)^{n-1}\right)}{(\sinh r)^{n-1}} d\mu^b,
\end{aligned}$$

since both integrals

$$\int_{S_{2r}} V\left(\varphi^{N-2}-1\right) d\mu^b \text{ and } \int_{S_r} V\left(\varphi^{N-2}-1\right) d\mu^b$$

converge to the same limit when $r \to \infty$. We know that $\varphi \leqslant 1 - \epsilon_0 e^{nr}$ for some $\epsilon_0 > 0$ so we have

$$H_\Phi^{\hat{g}}(V) = \lim_{r\to\infty} r^{-1} \int_{B_{2r}\setminus B_r} \frac{\varphi^{N-2}-1}{V} \frac{\nabla_r\left(V^2 (\sinh r)^{n-1}\right)}{(\sinh r)^{n-1}} d\mu^b < 0.$$

This contradicts the weak PMT, so we conclude that $\mathrm{Scal}^g \equiv -n(n-1)$.

Step 2. *g is an complete static metric:*

In this step we will follow the arguments of Dahl-Gicquaud-Sakovich in [Dahl *et al.*, 2014]. Let χ be an arbitrary smooth compactly supported function. We define the metric

$$g_u \coloneqq g + u\chi\left(\mathrm{Ric}^g - \frac{\mathrm{Hess}^g V^g}{V^g} + ng\right). \tag{6.25}$$

This is where the regularity assumption $e \in C^{4,\alpha}_\tau$ appears since the metrics g_u satisfy $e_u \coloneqq g_u - b \in C^{2,\alpha}_\tau$. Taking $u_0 > 0$ small enough, [Dahl *et al.*, 2014, Lemma 3.8] shows that for any $u \in [-u_0, u_0]$, there exists a unique positive functions φ_u on M s.t. the metric

$$\lambda_u \coloneqq \varphi_u^{\frac{4}{n-2}} g_u$$

has constant scalar curvature $-n(n-1)$. We also have by [Dahl *et al.*, 2014, Lemma 3.9 and 3.10] that the associated map $u \mapsto H(u) \coloneqq H_\Phi^{\lambda_u}(V)$ is a C^2 function and the derivative of H at $u = 0$ is given by

$$\dot{H}(0) = \int_M V^g \chi \left|\mathrm{Ric}^g - \frac{\mathrm{Hess}^g V^g}{V^g} + ng\right|^2 d\mu^g. \tag{6.26}$$

On the other hand, we know from the weak PMT for M that $H(s) \geqslant 0$ for all s. Combining this and the fact that $H(0) = 0$, we conclude that $\dot{H}(0) = 0$. Then, we obtain by (6.26) that

$$\chi \left|\mathrm{Ric}^g - \frac{\mathrm{Hess}^g V^g}{V^g} + ng\right| \equiv 0,$$

and, since χ is arbitrarily chosen, we get that

$$\mathrm{Ric}^g - \frac{\mathrm{Hess}^g V^g}{V^g} + ng \equiv 0.$$

This means that g is a complete static metric as claimed.

Step 3. *g is isometric to b:*

The idea of this last step is to use a result from [Qing, 2003]. Namely, (assuming that the positive mass theorem is true for asymptotically Euclidean manifolds) the hyperbolic space is the only complete static metric which is conformally compact with the round sphere as conformal infinity. The regularity required in his theorem does not appear clearly but since calculations are similar to the ones in [Lee, 1995], one might require that the metric g is $C^{3,\alpha}$-conformally compact for the arguments in [Qing, 2003] to work. However, we only know that $g - b \in C^{4,\alpha}_\tau$ but that suffices for the proof of [Qing, 2003] to work.

It should be noted that a similar result was proven for spin manifolds in [Wang, 2005]. The proof given there is based on the positive mass theorem for asymptotically hyperbolic manifolds so it is not suited for our purpose.

We set $\overline{g} := (V^g + 1)^{-2} g$. The scalar curvature of $\overline{g}$ is given by

$$\mathrm{Scal}^{\overline{g}} = n(n-1)\left((V^g)^2 - |dV^g|^2 - 1\right).$$

Some calculations show that the function $v := (V^g)^2 - |dV^g|^2_g - 1$ satisfies the following Bochner-type formula (see [Wang, 2005] for the derivation):

$$\Delta v = 2\left|\mathrm{Hess}(V^g) - V^g g\right|^2 - \left\langle \frac{dV}{V}, dv \right\rangle_g . \tag{6.27}$$

We are going to check that $v \to 0$ at infinity. It then follows from the strong maximum principle that $v \geqslant 0$ so the metric $\overline{g}$ has non-negative scalar curvature. Note that in the case $g = b$, we have $V^g \equiv \cosh r$ so $v \equiv 0$. From the proof of Lemma 6.4.4, we have $V^g - V = u \in C^{2,\alpha}_{\tau-1}$. Hence, $(V^g)^2 - V^2 = 2Vu + u^2 \in C^{2,\alpha}_{\tau-2}$. Similarly, we have

$$|dV^g|^2_g - |dV|^2_b \in C^{1,\alpha}_\tau .$$

Adding things up, we obtain that

$$v = (V^g)^2 - |dV^g|^2_g - 1 \in C^{1,\alpha}_{\tau-2}.$$

Since we are assuming that $n \geqslant 5$ and $\tau > n/2$, we have $\tau > 2$, so v tends to zero at infinity.

We next note that $\rho := (V+1)^{-1} = \frac{1-|x|^2}{2}$ is the standard defining function for the sphere of radius 1 in $\mathbb{R}^n$ and the metric $(V+1)^{-2} b$ is the Euclidean metric. In particular, using the diffeomorphism Φ, we can glue to M a sphere to get a smooth compact manifold with boundary $\overline{M}$.

From now on we assume that α is such that $2 + \alpha \leqslant \tau$. We check that $\overline{g}$ extends to a $C^{2,\alpha}$ metric on $\overline{M}$ and that $|\overline{g} - \delta|_\tau = O(\rho^2)$. Note that

$$\frac{V^g + 1}{V + 1} - 1 = \frac{u}{V+1} \in C^{2,\alpha}_\tau .$$

It follows from [Lee, 2006, Lemma 3.7] that $\frac{V^g+1}{V+1} \in C^{2,\alpha}(\overline{M})$ and that $\frac{V^g+1}{V+1} = 1 + O(\rho^\tau)$.

The tensor $e = g - b$ is twice covariant so it has rank 2 and we have $\rho^2 e \in C^{2,\alpha}_{\tau+2}$. Using once again [Lee, 2006, Lemma 3.7] we have that $\rho^2 e \in C^{2,\alpha}(\overline{M})$ and $|\rho^2 e|_\delta = |e|_b = O(\rho^\tau)$.

The metric $\overline{g}$ can be written as follows

$$\overline{g} = (V^g + 1)^{-2} g = \left(\frac{V^g + 1}{V+1}\right)^2 (\delta + \rho^2 e).$$

The previous arguments then show that $\overline{g}$ is a $C^{2,\alpha}$-metric on $\overline{M}$ and $|\overline{g} - \delta|_\tau = O(\rho^2)$. In particular $\overline{g}$ induces the round metric on the sphere $S^{n-1} = \overline{M} \setminus M$ and the second fundamental forms of S^{n-1} computed with respect to $\overline{g}$ and δ coincide.

The last part of the argument is to glue $\mathbb{R}^n \setminus B_1(0)$ (the outside of the unit ball in $\mathbb{R}^n$) to the manifold $\overline{M}$ so we get a new manifold $\widetilde{M}$ endowed with a metric $\widetilde{g}$ that coincide with $\overline{g}$ on $\overline{M}$ and with δ on $\mathbb{R}^n \setminus B_1(0)$. The metric $\widetilde{g}$ is smooth on $\widetilde{M} \setminus S_1(0)$ and is globally C^2.

From the rigidity part of the positive mass theorem for asymptotically Euclidean manifolds, the manifold $(\widetilde{M}, \widetilde{g})$ is isometric to the Euclidean space, meaning that the metric $\overline{g}$ is flat. The metric g is then conformally equivalent to the hyperbolic metric $g = \phi^{N-2} b$ with $e = (\phi^{N-2} - 1) b \in C^{2,\alpha}_\tau$ so $\phi - 1 \in C^{2,\alpha}_\tau$. Since both g and b have scalar curvature $-n(n-1)$, we conclude by the uniqueness of the solution to the Yamabe equation

$$\frac{4(n-1)}{n-2} \Delta^b \phi - n(n-1)\phi = -n(n-1)\phi^{N-1}$$

(see e.g. [Gicquaud, 2010]) that $\phi \equiv 1$ and $g \equiv b$. □

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